Practice problem set 1

1. Let G be a graph with n vertices and $e(G)$ edges, and set $k = \lfloor n/2 \rfloor$. Show that there exists a cut of the graph G containing at least $\frac{k}{2k-1}e(G)$ edges.
(Hint: consider only those cuts $(A, V \setminus A)$, where $|A| = k$, and choose one of them uniformly at random. Finally count the expected size of $e(A, V \setminus A)$. It is worth distinguishing the cases, where $n = 2k$ or $n = 2k + 1$.)

2. A set system $\mathcal{H}$ is called Sperner if for different $A, A' \in \mathcal{H}$ neither $A \subseteq A'$, nor $A' \subseteq A$. Show that for a Sperner-system $\mathcal{H}$ on $\{1, 2, \ldots, n\}$ we have

$$\sum_{A \in \mathcal{H}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

(Hint: consider a random permutation of $1, 2, \ldots, n$ and for an $A \in \mathcal{H}$ let E_A be the event that the first $|A|$ elements of the permutation are exactly the elements of A. What is $\mathbb{P}(E_A)$? Show that if $A \not\subseteq A'$ and $A' \not\subseteq A$ then $\mathbb{P}(E_A \cap E_{A'}) = 0$. From that conclude that if $\mathcal{H}$ is Sperner then $\sum_{A \in \mathcal{H}} \mathbb{P}(E_A) \leq 1$.)

3. Show that for any n positive integers one can choose $\lfloor n/3 \rfloor$ of them such that the equation $a_1 + a_2 = a_3$ has no solution.
(Hint: let A be the given set, and for an $x \in (0, 1)$ consider the set $A_x = \{a \in A \mid \{ax\} \in (1/3, 2/3)\}$, where $\{ax\}$ is the fractional part of ax, that is, $ax - \lfloor ax \rfloor$. Show that if $a_1, a_2, a_3 \in A_x$ then $a_1 + a_2 \neq a_3$, and compute $\mathbb{E}|A_x|$ if we choose $x \in (0, 1)$ uniformly at random.)