Problem set 1

Due date: 02/29/2024 4pm

Introduction to Tutte polynomial

Before you start working on the homework problems read the extended syllabus[1] carefully and check which problems you need to solve.

Practice exercises:

1. Let $\mathrm{ch}(G, q)$ be the chromatic polynomial. Show that $|\mathrm{ch}(G, -1)|$ is equal to the number of acyclic orientations of the graph G.

2. Let $F(G)$ be the number of forests of a graph G, and let $a(G)$ denote the number of acyclic orientations of the graph G. Show that $F(G) \geq a(G)$.

4. Show that if $B_1, \dots, B_k$ are the 2-connected blocks of a graph G, then

$$T_G(x, y) = \prod_{i=1}^{k} T_{B_i}(x, y).$$

Homework exercises:

1. (a) Show that $T_{C_k}(x, y) = x^{k-1} + x^{k-2} + \cdots + x + y$.
(b) Let $K_2(k)$ be the graph with 2 vertices and k edges between them. Show that $T_{K_2(k)}(x, y) = y^{k-1} + y^{k-2} + \cdots + y + x$.

2. Let $\tau(G)$ denote the number of spanning trees of a graph G. Suppose that G has at least 2 vertices. Show that

$$\tau(G) \leq \prod_{v \in V(G)} d_v,$$

where d_v denotes the degree of a vertex v.

3. Show that the chromatic polynomial satisfies the identity

$$\sum_{S \subseteq V(G)} \mathrm{ch}(G[S], x)\mathrm{ch}(G[V \setminus S], y) = \mathrm{ch}(G, x + y).$$

Here $G[S]$ is the induced subgraph of the graph G on the vertex set S. Note that $\mathrm{ch}(G[\emptyset], x) = 1$.

4. Let G be a 4–regular graph. Show that the number of Eulerian orientations is exactly $|T_G(0, -2)|$. (An Eulerian orientation is just an orientation of the edges where the in-degree is equal to the out-degree at every vertex of the graph.)

5. Let G be a d–regular graph on n vertices. Let $T_G(x, y)$ denote the Tutte polynomial of the graph G. Show that $T_G(0, 1) \leq (d - 1)^n$.

6. Let G be a graph and for each orientation $\mathcal{O}$ of the graph G let $d_{\mathcal{O}}(v)$ be the out-degree of the vertex v. Consider the vectors $d_{\mathcal{O}} = (d_{\mathcal{O}}(v_1), \dots, d_{\mathcal{O}}(v_n)) \in \mathbb{R}^n$. Let

[1] The extended syllabus can be found at
http://csikvarip.web.elte.hu/extended_syllabus_Tutte_polynomial.pdf

P be a polytope determined by the vectors $d_{\mathcal{O}}$. Show that the extreme points of this polytope are exactly the vectors coming from acyclic orientations. In other words, if $\mathcal{O}$ is not an acyclic orientation then it can be written as a convex combination of other vectors $d_{\mathcal{O}'}$, while if $\mathcal{O}$ is an acyclic orientation then it cannot be written as a convex combination of other vectors $d_{\mathcal{O}'}$.

7. Let G be a graph and for each vertex u we introduce a variable x_u. Let us consider the polynomial

$$P := \prod_{(u,v)\in E(G)} (x_u + x_v).$$

First show that with the notations of the previous problem we have

$$P = \sum_{\mathcal{O}} \prod_{u\in V(G)} x_u^{d_{\mathcal{O}}(u)}.$$

Suppose that we expand P in the usual way:

$$P = \sum c_{\underline{e}} \prod_{u\in V(G)} x_u^{e_u}.$$

Show that the number of acyclic orientations is equal to the number of terms with coefficient 1, that is, $c_{\underline{e}} = 1$.

8. Let G be a simple graph. Let $T_G(x,y)$ denote the Tutte polynomial of the graph G. Show that $T_G(1,-1) \geq 0$.

9. For a graph G and an orientation $\mathcal{O}$ let $d_{\mathcal{O}}$ be the out-degree sequence. Show that the number of different out-degree sequences is exactly $T_G(2,1)$, the number of forests.

10. Let G be a d–regular bipartite graph. Let $T_G(x,y)$ denote the Tutte polynomial of the graph G. Show that $T_G(-d+1,1) = 0$.