Problem set 2

Due date: 03/21/2024 4pm
Permutation Tutte polynomial

Before you start working on the homework problems read the extended syllabus[1] carefully and check which problems you need to solve.

In all problems $\widetilde{T}_H(x,y)$ denotes the permutation Tutte polynomial while $T_G(x,y)$ denotes the usual Tutte polynomial.

1. Let $H=(A,B,E)$ be a connected bipartite graph with $|A|=a$ and $|B|=b$. Let $\mathrm{alt}(H)$ be the coefficient of x^a. Show that this is also the coefficient of y^b.

2. Determine $\widetilde{T}_{K_{a,b}}(x,y)$ for the complete bipartite graph $K_{a,b}$. Determine the Tutte polynomial of the uniform matroid $U_{n,r}$ whose bases are all the r-element subsets of the n-element ground set.

3. Suppose that G is a 2-connected loopless graph and it is not a cycle. Let $T_G(x,y)=\sum_{i,j} t_{i,j}x^iy^j$ be its Tutte polynomial. Show that $t_{1,1}\geq 1$.

4. Let G be a 2-connected loopless graph on n vertices whose shortest cycle has length g. Let $T_G(x,y)=\sum_{i,j} t_{i,j}x^iy^j$ be its Tutte polynomial. Show that $t_{i,1}\geq 1$ for $0\leq i\leq n-g$.

5. Suppose that H is a connected bipartite graph and $\widetilde{T}_H(x,y)=\sum t_{i,j}x^iy^j$. Show that if $t_{i,j}>0$ for some i and j, then $t_{i',j'}>0$ whenever $i'\leq i, j'\leq j$ and $(i',j')\neq(0,0)$. Conclude that the same statement holds true for the Tutte polynomial $T_G(x,y)$ of every 2-connected graph G.

6. Let G be a connected planar graph with dual graph G^*. Show that
$T_G(x,y)=T_{G^*}(y,x)$.

7. Let $H=(A,B,E)$ be a connected bipartite graph with $|A|=a$ and $|B|=b$. Let $\mathrm{alt}(H)$ be the coefficient of x^a. Show that for $(x-1)(y-1)=1$ we have

$$\widetilde{T}_H(x,y)=\mathrm{alt}(H)x^ay^b.$$

8. Let $H=(A,B,E)$ be a connected bipartite graph. For $S\subseteq A$ or $S\subseteq B$ let H_S be the subgraph of H induced by the set $S\cup N_H(S)$, where $N_H(S)=\{u\in V\mid \exists\, v\in S: (u,v)\in E\}$. Show that

$$\widetilde{T}_H(x,y)=\sum_{\substack{S\subseteq A, T\subseteq B\\ e(S,T)=0}}\mathrm{alt}(H_S)\mathrm{alt}(H_T)(x-1)^{|S|}(y-1)^{|T|}.$$

9. Show that for every bipartite graph H there is a matroid M and a basis $B\in\mathcal{B}(M)$ for which the local basis exchange graph $H_M[B]$ is isomorphic to H.

[1]The extended syllabus can be found at
`http://csikvarip.web.elte.hu/extended_syllabus_counting_in_sparse_graphs.pdf`

10. Let $T_G(x,y)$ be the Tutte polynomial of a graph G.
(a) Show that

$$T_G(2,1)T_G(1,2) \geq T_G(1,1)T_G(2,2).$$

(b) For parameters $q > 1$ and $w > 0$ we define the following probability space on the subsets of the edge set $E(G)$. For an $F \subseteq E(G)$ let

$$\mathbb{P}_{q,w}(F) = \frac{1}{Z} q^{k(F)} w^{|F|},$$

where Z is the normalizing constant: $Z = \sum_{F \subseteq E(G)} q^{k(F)} w^{|F|}$. Show that

$$\mathbb{P}_{q,w}(F_1) \cdot \mathbb{P}_{q,w}(F_2) \leq \mathbb{P}_{q,w}(F_1 \cap F_2) \cdot \mathbb{P}_{q,w}(F_1 \cup F_2).$$

(c) Show that if $1 \leq x_1 < x_2$, $1 \leq y_1 < y_2$ and $(x_2 - 1)(y_2 - 1) \geq 1$, then

$$T_G(x_1,y_2)T_G(x_2,y_1) \geq T_G(x_1,y_1)T_G(x_2,y_2).$$

11. (a) Let $H = (A,B,E)$ be a connected bipartite graph with $|A| = a$ and $|B| = b$. Let $\mathrm{alt}(H)$ be the coefficient of x^a. Show that $\mathrm{alt}(H) \geq \left(\frac{1}{2}\right)^{e(H)}$.
(b) For a connected graph G let us consider the graph $\mathcal{T}_G$ whose vertices are the spanning trees of G and we connect two vertices if the symmetric difference of the two spanning trees has size exactly 2. Show that the largest degree of $\mathcal{T}_G$ is at least $\log_2 \tau(G)$, where $\tau(G)$ is the number of spanning trees.

12. Let G be a connected graph, and let us put a weight to each edge selected uniformly at random from the interval $[0,1]$. Let W_G be the weight of the minimum weight spanning tree. Show that the expected value $\mathbb{E}W_G$ of the random variable W_G is determined by the Tutte polynomial of the graph G.