Problem set 3

Due date: 04/18/2024 4pm

Matching and subgraph counting polynomial

Before you start working on the homework problems read the extended syllabus[1] carefully and check which problems you need to solve.

In all problems $\mu(G,x)=\sum_{k=0}^{n/2}(-1)^k m_k(G)x^{n-2k}$, where G is a graph on n vertices and $m_k(G)$ denotes the number of matchings of size k. Furthermore, for a d-regular graph G let $F_G(x_0,\dots,x_d)=\sum_{A\subseteq E}\left(\prod_{v\in V}x_{d_A(v)}\right)$ be the subgraph counting polynomial.

1. Let $G=(A,B,E)$ be a bipartite graph on n vertices. Show that

$$\sum_{u\in A}\mu(G-u,x)=\frac{1}{2}\frac{d}{dx}\mu(G,x)+\left(|A|-\frac{n}{2}\right)\frac{\mu(G,x)}{x}.$$

2. Let G be a 6-regular graph. Express $F_G(1,0,1,0,1,0,1)$ in terms of the number of vertices and connected components of G.

3. Let $\widetilde{M}(G,x)=x^n+m_1(G)x^{n-1}+m_2(G)x^{n-2}+\dots$. Show that

$$\sum_{S\subseteq V(G)}\widetilde{M}(G[S],x)\widetilde{M}(G[V-S],y)=\widetilde{M}(G,x+y).$$

Here $G[S]$ is the induced subgraph on the vertex set S.

4. Let G be a d-regular graph and for some β let $\gamma=\frac{e^\beta-e^{-\beta}}{e^\beta+e^{-\beta}}$. Show that

$$\sum_{\sigma\in\{-1,1\}^{V(G)}}\exp\left(\beta\sum_{(u,v)\in E(G)}\sigma_u\sigma_v\right)=2^{v(G)}\left(\frac{e^\beta+e^{-\beta}}{2}\right)^{e(G)}F_G(1,0,\gamma,0,\gamma^2,0,\dots).$$

5. Let G be a graph and let $u\in V(G)$ be a fixed vertex. Let λ be a root of the matching polynomial $\mu(G,x)$ such that $\mu(G-u,\lambda)\neq 0$. We know that λ is an eigenvalue of the adjacency matrix of the path-tree $T(G,u)$. Give a corresponding eigenvector of the matrix! (It is enough to give one vector, you don't have to determine the whole eigenspace.) Give the answer in terms of the numbers $\mu(H,\lambda)$, where H are induced subgraphs of G.

6. (a) Let $\gamma(G)$ be the largest zero of the matching polynomial of a graph G. Show that if H is a subgraph of G, then $\gamma(H)\leq\gamma(G)$.

(b) Show that if $T(G,u)$ is the path-tree of the connected graph G from vertex u, then $\gamma(T(G,u))=\gamma(G)$.

[1] The extended syllabus can be found at
http://csikvarip.web.elte.hu/extended_syllabus_counting_in_sparse_graphs.pdf

7. Let v be a vertex of G, and

$$\frac{\mu(G-v,z)}{\mu(G,z)} = \sum_k \frac{c_k}{z-\alpha_k}.$$

Show that $c_k \geq 0$.

8. Let G be a 4-regular graph. Show that $F_G(0,1,0,0,0) = F_G\left(1,-\frac{1}{2},0,\frac{1}{2},-1\right)$. (Hint: use $G_e = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$ at every edge with appropriate t.)

9. Let G be a d-regular graph. Let $\gamma(G)$ be the largest zero of the matching polynomial $\mu(G,x)$. Show that $\gamma(G) \geq \gamma(K_{d+1})$. Show that if G is bipartite, then $\gamma(G) \geq \gamma(K_{d,d})$ is also true.

10. Given a graph G. Let $m_k(G)$ denote the number of matchings of size k. Show that the numbers $m_1(G), m_2(G), \dots, m_{\lfloor n/2 \rfloor}(G)$ together with the number of vertices determine the numbers $m_1(\overline{G}), m_2(\overline{G}), \dots, m_{\lfloor n/2 \rfloor}(\overline{G})$, where $\overline{G}$ is the complement of the graph G.