Problem set 4

Due date: 05/09/2024 4pm

Gauge transformation

Before you start working on the homework problems read the extended syllabus[1] carefully and check which problems you need to solve.

In all problems $F_G(x_0, x_1, \dots, x_d)$ is the subgraph counting polynomial of a d-regular graph G.

1. Let G be a d-regular graph and let γ be a fixed number. What is $F_G(1, \gamma, \gamma^2, \dots, \gamma^d)$?

2. (Coding problem.) Let G be the graph on 6 that we obtained from the complete graph by deleting the edges of a perfect matching. Write a computer code and determine the polynomial $P_G(z) = F_G\left(\frac{3}{2}, 0, -\frac{1}{2}z, 0, \frac{3}{2}z^2\right)$, plot its roots on the complex plane.

3. Let G be 4-regular graph. Show that $F_G(0,0,1,0,0) = F_G\left(\frac{3}{2}, 0, -\frac{1}{2}, 0, \frac{3}{2}\right)$. Conclude that G has more Eulerian orientations than 2-regular subgraphs.

4. For a fixed $\lambda > 0$ and a graph G on n vertices let $M_G(\lambda) = \sum_{k=0}^{\lfloor v(G)/2 \rfloor} m_k(G)\lambda^k$, where $m_k(G)$ denotes the number of k-matchings. For a 3-regular graph G let G^Δ be the graph that we obtain from G replacing each vertex u of G with neighbors x, y, z by a triangle with vertices u_x, u_y, u_z and if $(u, v) \in E(G)$, then we connect u_v with v_u. Thus the resulting graph has $3n$ vertices and is also 3-regular. Show that $M_{G^\Delta}(\lambda) = F_G(1 + 3\lambda, (1 + \lambda)\lambda^{1/2}, \lambda, \lambda^{3/2})$.

5. Let C_n be the cycle on n vertices. For $a_0, a_1, a_2 \in \mathbb{R}_{\geq 0}$ let

$$\Psi(a_0, a_1, a_2) = \lim_{n \to \infty} F_{C_n}(a_0, a_1, a_2)^{1/n}.$$

Show hat

$$\Psi(a_0, a_1, a_2) = \max_{t \in [0, 2\pi]} (a_0 \cos(t)^2 + 2a_1 \cos(t)\sin(t) + a_2 \sin(t)^2).$$

Show that for even n we always have $F_{C_n}(a_0, a_1, a_2)^{1/n} \geq \Psi(a_0, a_1, a_2)$, while for odd n we have $F_{C_n}(a_0, a_1, a_2)^{1/n} \geq \Psi(a_0, a_1, a_2)$ if and only if $a_0 a_2 - a_1^2 \geq 0$.

6. Let K_4 be the complete graph on 4 vertices. Show that for arbitrary $(a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ we have $F_{K_4}(a_0, a_1, a_2, a_3) \geq 0$.

7. Let G be a d-regular graph. By using the gauge transformation $G_e = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$ everywhere show that there are functions $\widehat{a_0}(t), \widehat{a_1}(t), \dots, \widehat{a_d}(t)$ on $[0, 2\pi]$ for which $F_G(a_0, a_1, \dots, a_d) = F_G(\widehat{a_0}(t), \widehat{a_1}(t), \dots, \widehat{a_d}(t))$, and $\widehat{a_0}(t) = \sum_{k=0}^{d} a_k \binom{d}{k} \cos(t)^{d-k} \sin(t)^k$. Let $t_0 \in [0, 2\pi]$ for which $\widehat{a_0}(t)$ is maximal. Show that $\widehat{a_1}(t_0) = 0$.

[1] The extended syllabus can be found at
http://csikvarip.web.elte.hu/extended_syllabus_counting_in_sparse_graphs.pdf

8. Let $K_2(d)$ be the graph on 2 vertices with d parallel edges between the two vertices. Let G be an arbitrary bipartite d-regular graph on $2n$ vertices. Show that for arbitrary $(a_0, a_1, \dots, a_d) \in \mathbb{R}^{d+1}$ we have $F_{K_2(d)}(a_0, a_1, \dots, a_d)^n \geq F_G(a_0, a_1, \dots, a_d)$.

9. Let G be a connected 3–regular graph on n vertices. Let us choose an orientation $\mathcal{O}$ of G uniformly at random. Let $n_+(\mathcal{O})$ be the number of vertices with out-degree 3. and let $n_-(\mathcal{O})$ be the number of vertices with in-degree 3. Show that the probability of $n_+(\mathcal{O}) - n_-(\mathcal{O}) = k$ is $\frac{\binom{n}{n/2-2k}}{2^{n-1}}$.

10. Let G be a 3-regular graph, and $a_0, a_1, a_2, a_3 \geq 0$. Let

$$\Psi(a_0, a_1, a_2, a_3) = \max_{t \in [0, 2\pi]} \sum_{k=0}^{3} a_k \binom{3}{k} \cos(t)^{3-k} \sin(t)^k.$$

Show that if $a_0 a_2 + a_1 a_3 - a_1^2 - a_2^2 \geq 0$, then for every 3-regular graph G on n vertices we have

$$F_G(a_0, a_1, a_2, a_3) \geq \Psi(a_0, a_1, a_2, a_3)^n.$$

(By the way, is this condition satisfied for the vector given in Problem 4?)