Problem set 5

Due date: 05/30/2024 4pm

Empirical measures and limits

Before you start working on the homework problems read the extended syllabus[1] carefully and check which problems you need to solve.

1. Let $x > 1$ and let $(G_n)_n$ be a sequence of connected d-regular graphs such that $v(G_n) \to \infty$. Compute the limit

$$\lim_{n\to\infty} T_G\left(x, \frac{x}{x-1}\right)^{1/v(G_n)}.$$

2. For a graph G on n vertices let $\mathrm{ch}(G,q) = \prod_{j=1}^{n}(q-\alpha_i)$ be the chromatic polynomial. Let $\rho_G^c = \frac{1}{n}\sum_{j=1}^{n}\delta_{\alpha_j}$ be the uniform measure supported on the zeros of $\mathrm{ch}(G,q)$, where we take the multiplicities into account. Show that the sequence of paths and cycles are Benjamini–Schramm convergent together, but the sequence of measures $\rho_{P_n}^c, \rho_{C_n}^c$ is not weakly convergent together.

3. Let $d \geq 3$ and let $\mathbb{T}_d^k$ be the finite graph obtained from the infinite d-regular tree $\mathbb{T}_d$ by keeping the vertices that are of distance at most k from a fixed vertex o. Let $0 < p < \frac{1}{d-1}$ and let us consider a random subgraph of $\mathbb{T}_d^k$ by keeping each edge with probability p and delete it with probability $1-p$. Let p_k be the probability that there is a path from o to a vertex of distance k in the obtained random graph. Show that $\lim_{k\to\infty} p_k = 0$. For bonus 1 point: is the statement still true if $p = \frac{1}{d-1}$?

4. Let G be a d–regular graph on n vertices with chromatic polynomial $\mathrm{ch}(G,x) = \prod_{i=1}^{n}(x-\alpha_i)$. Let g be the length of the shortest cycle of the graph G. Determine the value of

$$\frac{1}{n}\sum_{i=1}^{n}\alpha_i^k$$

for $1 \leq k \leq g-2$.

5. Let G be a graph. Show that the chromatic polynomial $\mathrm{ch}(G,x)$ has no zero in the open interval $(0,1)$.

6. Let G be a graph. Show that the matching polynomial $\mu(G,x)$ has a zero in the closed interval $[0,1]$.

7. Let G be a bipartite graph that is not a forest. Show that its chromatic polynomial has a non-real zero.

[1]The extended syllabus can be found at
http://csikvarip.web.elte.hu/extended_syllabus_counting_in_sparse_graphs.pdf

8. Let q be a positive integer and $-1 \leq w \leq 0$. As usual let $Z_G(q,w) = \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|}$, where $k(A)$ denotes the number of connected components of the graph (V,A). Show that if G is a bipartite graph with $v(G)$ vertices and $e(G)$ edges, then

$$Z_G(q,w) \geq q^{v(G)} \left(1 + \frac{w}{q}\right)^{e(G)}.$$

Is the statement true if $q > 2$, but not necessarily integer?

9. Let $F_G(x_0, x_1, \ldots, x_d)$ be the subgraph counting polynomial. Suppose that $0 < \gamma < \frac{1}{d-1}$. Show that if $(G_n)_n$ is a sequence of d-regular graphs such that $g(G_n) \to \infty$, then

$$\lim_{n \to \infty} F_{G_n}(1, 0, \gamma, 0, \gamma^2, 0, \gamma^3, \ldots)^{1/v(G_n)} = 1.$$

Let $\beta > 0$ such that $\frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}} < \frac{1}{d-1}$ and

$$Z_{\mathrm{Is}(\beta)}(G) = \sum_{\sigma : V(G) \to \{-1,1\}} \exp\left(\beta \sum_{(i,j) \in E(G)} \sigma_i \sigma_j\right).$$

Show that if $(G_n)_n$ is a sequence of d-regular graphs such that $g(G_n) \to \infty$, then

$$\lim_{n \to \infty} Z_{\mathrm{Is}(\beta)}(G)^{1/v(G_n)} = 2\left(\frac{e^\beta + e^{-\beta}}{2}\right)^{d/2}.$$

10. For a graph G let ssfo(G) be the number of orientations of G without source and sink. (SSfo stands for source and sink free orientation.) Let $(G_n)_n$ be a sequence of 3-regular graphs such that $g(G_n) \to \infty$, where $g(G)$ denotes the girth of the graph G. Show that

$$\lim_{n \to \infty} \mathrm{ssfo}(G_n)^{1/v(G_n)} = \frac{3}{\sqrt{2}}.$$