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Tutte polynomial

Lecture note

Contents

Preface

Tutte polynomial is a fascinating object that unifies several counting problems about graphs including the enumeration of proper colorings, spanning trees and acyclic orientations.

This lecture note consists of two parts. In the first part we introduce the basics of the Tutte polynomial. Even in this part we included recent advancements like the refutation of the Merino–Welsh conjecture for matroids.

The second part of the lecture note is about limits of Tutte polynomials. This part is admittedly much more involved and the tools introduced here go way beyond the study of the Tutte polynomial. It turns out that it is very hard to work with the Tutte polynomial directly, and we need to introduce concepts like the matching polynomial that are much more amenable to study. Building a bridge between the Tutte polynomial and the matching polynomial also requires new tools like the gauge transformation which is essentially a combinatorial identity prover technique that also provides unexpected connections between subgraphs and orientations. Another useful concept that we will utilize is the empirical root measure that constitute a bridge between finite graphs and their limit objects. Here we also need to develop the basics of the Benjamini–Schramm convergence to make our graph convergence arguments rigorous.

This is a hard course that involves ideas from several branches of mathematics. If you understand the first part and have a superficial understanding of the second part that is already a very good achievement. We will not be able to cover everything at the classes. Parts that are omitted at the lectures are marked with a *.

I hope you will you enjoy the course!

Part I

Basics

1. Tutte polynomial

1.1 Introduction

In this chapter we introduce the so-called Tutte polynomial.

Definition 1.1.1 ([48]). Let $G = (V, E)$ be an arbitrary graph. Then the Tutte polynomial of G is defined as

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|},$$

where $k(A)$ denotes the number of connected components of the graph (V, A).

There are many excellent surveys about the properties of the Tutte polynomial and its applications [9, 21, 56, 24] or the book [25].
In this lecture note we allow loops and multiple edges. The Tutte polynomial satisfies the following recursion:

$$T_G(x, y) = \begin{cases} xT_{G-e}(x, y) & \text{if } e \in E(G) \text{ is a bridge,} \\ yT_{G-e}(x, y) & \text{if } e \in E(G) \text{ is a loop,} \\ T_{G-e}(x, y) + T_{G/e}(x, y) & \text{if } e \in E(G) \text{ is neither a bridge, nor a loop.} \end{cases}$$

Bridge is just another word for a cut edge.

From this recursion it follows that

$$T_G(x, y) = \sum_{i,j} t_{i,j} x^i y^j,$$

where $t_{i,j} \geq 0$. These numbers have a combinatorial meaning in terms of spanning trees.

1.2 Special points

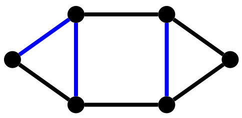
The main feature of the Tutte polynomial is the wide variety of enumeration problems that are special evaluations of the Tutte polynomial.

Theorem 1.2.1. *Let G be a connected graph.*
(a) $T_G(1,1)$ counts the number of spanning trees.
(b) $T_G(2,1)$ counts the number of spanning forests, that is, acyclic edge subsets.
(c) $T_G(1,2)$ counts the number of connected subgraphs.
(d) $T_G(2,2) = 2^{e(G)}$.

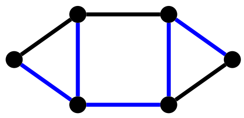
Proof. We have

$$T_G(x,y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|},$$

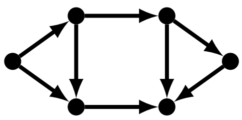
where $k(E) = 1$ since G is connected. Note that $0^k = 0$ except if $k = 0$, thus $(1-1)^{k(A)-k(E)}(1-1)^{k(A)+|A|-|V|} = 1$ if and only if $k(A) = k(E) = 1$, and $|A| = |V| - k(A) = |V| - 1$, thus A is the edge set of a spanning tree. Parts (b) and (c) follow the same way, part (d) is completely trivial. $\square$



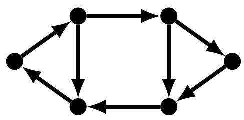
(a) $T_G(2,1)$ spanning forests



(b) $T_G(1,1)$ spanning trees



(c) $T_G(2,0)$ acyclic orientations



(d) $T_G(0,2)$ strong orientations

Theorem 1.2.2. *Let G be a connected graph.*
(a) $T_G(2,0)$ counts the number of acyclic orientations.
(b) $T_G(1,0)$ counts the number of acyclic orientations with a unique source vertex that is fixed.
(c) $T_G(0,2)$ counts the strongly connected orientations.

Proof. We only prove part (a), the other two parts are similar.

Let $a(G)$ be the number of acyclic orientations. If G contains a loop, then $a(G) = 0$. If G contains a bridge e, then $a(G) = 2a(G-e)$ since we can direct the edge e anyway. If an edge e is neither a loop, nor a bridge, then let us consider an acyclic orientation $\mathcal{O}$ of the graph $G-e$. Such an orientation always comes from a topological order, that is, an ordering of the vertices in such a way that every edge is oriented according to the orientation, left to right. Thus we can always orient the edge e at least one way: simply orient it according to some topological order consistent with the acyclic orientation $\mathcal{O}$. It can happen that we can orient the edge e in both ways. This means that in $G-e$ there is no directed path between the end vertices of e in any direction. Then it means that $\mathcal{O}$ corresponds to an acyclic orientation of G/e. Hence $a(G) = (a(G-e) - e(G/e)) + 2a(G/e) = a(G-e) + a(G/e)$. For the empty graph O_n on n vertices we have $a(O_n) = T_{O_n}(2,0) = 1$ (empty edge set A), thus the recursion and the base cases of $a(G)$ and $T_G(2,0)$ coincide. Hence $T_G(2,0) = a(G)$.

□

1.3 Chromatic polynomial

In this section we introduce the so-called chromatic polynomial.

Definition 1.3.1. [45] Let G be a graph. A map $\varphi : V(G) \to \{1,2,\dots,q\}$ is a proper coloring with q colors if $\varphi(u) \neq \varphi(v)$ whenever $(u,v) \in E(G)$.

The number of proper colorings of G with q colors is denoted by $\mathrm{ch}(G,q)$.

Remark 1.3.2. The function $\mathrm{ch}(G,q)$ is polynomial in q. This is called the chromatic polynomial of the graph G.

Proposition 1.3.3. *If $e \in E(G)$ is a loop, then $\mathrm{ch}(G,q) = 0$. If $e \in E(G)$ is a bridge, then $\mathrm{ch}(G,q) = \frac{q-1}{q}\mathrm{ch}(G-e,q)$. If $e \in E(G)$ is neither loop, nor bridge, then we have*

$$\mathrm{ch}(G,q) = \mathrm{ch}(G-e,q) - \mathrm{ch}(G/e,q),$$

where G/e denotes the graph obtained from G by contracting the edge e.

Proof. Let us consider the proper colorings of $G-e$. If $e = (u,v)$ then we can distinguish two cases: if u and v get different colors then it is even a proper coloring of G. If u and v get the same color then it corresponds to a proper coloring of G/e. Hence

$$\mathrm{ch}(G-e,q) = \mathrm{ch}(G,q) + \mathrm{ch}(G/e,q).$$

□

The chromatic polynomial is a special evaluation of the Tutte-polynomial. Indeed, one only needs to compare the recursion formulas of the two polynomials to prove the following statement.

Theorem 1.3.4. *We have*

$$\mathrm{ch}(G,q) = (-1)^{v(G)-k(G)} q^{k(G)} T_G(1-q,0).$$

1.4 Statistical physics and the Tutte polynomial

In this section we review some classical statistical physical models that are related to the Tutte polynomial.

1.4.1 Ising-model

In the case of the Ising-model the vertices of the graph G represent particles. These particles have a spin which can be up $(+1)$ or down (-1). Two adjacent particles have an interaction e^{β} if they have the same spin, and $e^{-\beta}$ if they have different spin. Suppose also that there is an external magnetic field that breaks the symmetry between $+1$ and -1. This defines a probability distribution on the possible configurations as follows: for a random spin configuration $\mathbf{S}$:

$$\mathbb{P}(\mathbf{S}=\sigma) = \frac{1}{Z}\exp\left(\sum_{(u,v)\in E(G)} \beta\sigma(u)\sigma(v) + B\sum_{u\in V(G)}\sigma(u)\right),$$

where Z is the normalizing constant:

$$Z_{\mathrm{Is}}(G,B,\beta) = \sum_{\sigma:V(G)\to\{-1,1\}} \exp\left(\sum_{(u,v)\in E(G)} \beta\sigma(u)\sigma(v) + B\sum_{u\in V(G)}\sigma(u)\right).$$

Z is called the partition function of the Ising-model.

When $\beta > 0$ we say that it is a ferromagnetic Ising-model, and when $\beta < 0$, then we say that it is an antiferromagnetic model. It turns out that the model behaves very differently in the two regimes even if we only consider extremal graph theoretic questions.

1.4.2 Potts-model and random-cluster model

Potts-model is a generalization of the Ising-model. Again the vertices of the graph G represent particles, but this time their spin or state can be one of q different states, i. e., $\sigma : V(G) \to [q]$. Two adjacent particles have an interaction e^{β} if they are in the same state, and no interaction otherwise. This defines a probability distribution on the possible configurations as follows: for a random configuration $\mathbf{S}$: let $\mathbf{1}(\text{statement}) = 1$ if the statement is true and 0 otherwise, then

$$\mathbb{P}(\mathbf{S} = \sigma) = \frac{1}{Z} \exp\left(\sum_{(u,v)\in E(G)} \beta \mathbf{1}(\sigma(u) = \sigma(v)) \right),$$

where Z is the normalizing constant:

$$Z_{\mathrm{Po}}(G, q, \beta) = \sum_{\sigma:V(G)\to[q]} \exp\left(\sum_{(u,v)\in E(G)} \beta \mathbf{I}(\sigma(u) = \sigma(v)) \right).$$

Similarly to the previous case, Z is called the partition function of the Potts-model. In the case when β is large, then the system prefers configurations where the particles are in the same state. When β is a large negative number, then the system prefers configuration where adjacent vertices are in different states. In the limiting case $\beta = -\infty$ we get that $\lim_{\beta\to-\infty} e^{-\beta|E|} Z = \mathrm{ch}(G, q)$.

Potts-model is very strongly related to the so-called random cluster model. In this model, the probability distribution is on the subsets of the edge set $E(G)$, and for a random subset $\mathbf{F}$ we have

$$\mathbb{P}(\mathbf{F} = F) = \frac{1}{Z} q^{k(F)} w^{|F|},$$

where $k(F)$ is the number of connected components of the graph $G' = (V(G), F)$. Here q is non-negative and $w \geq -1$, but not necessarily integers, and Z is the normalizing constant

$$Z_{\mathrm{RC}}(G, q, w) = \sum_{F\subseteq E(G)} q^{k(F)} w^{|F|}.$$

Lemma 1.4.1. *Let q be a positive integer, and $e^{\beta} = 1 + w$. Then*

$$Z_{\mathrm{Po}}(G, q, \beta) = Z_{\mathrm{RC}}(G, q, w).$$

Proof. Clearly,

$$
\begin{aligned}
Z_{\mathrm{Po}}(G,q,\beta) &= \sum_{\sigma:V(G)\to[q]} \exp\left(\sum_{(u,v)\in E(G)} \beta\mathbf{I}(\sigma(u)=\sigma(v))\right) \\
&= \sum_{\sigma:V(G)\to[q]} \prod_{(u,v)\in E(G)} (1+(e^{\beta}-1)\mathbf{I}(\sigma(u)=\sigma(v))) \\
&= \sum_{\sigma:V(G)\to[q]} \prod_{(u,v)\in E(G)} (1+w\mathbf{I}(\sigma(u)=\sigma(v))) \\
&= \sum_{\sigma:V(G)\to[q]} \sum_{A\subseteq E(G)} w^{|A|} \prod_{(u,v)\in A} \mathbf{I}(\sigma(u)=\sigma(v)) \\
&= \sum_{A\subseteq E(G)} w^{|A|} \left(\sum_{\sigma:V(G)\to[q]} \prod_{(u,v)\in A} \mathbf{I}(\sigma(u)=\sigma(v))\right) \\
&= \sum_{A\subseteq E(G)} w^{|A|} q^{k(A)} \\
&= Z_{\mathrm{RC}}(G,q,w).
\end{aligned}
$$

□

There is a very clear connection between the Tutte-polynomial and the partition function of the random cluster model $Z_{\mathrm{RC}}(G,q,w)$. Namely,

$$T_G(x,y) = (x-1)^{-k(G)}(y-1)^{-v(G)} Z_{\mathrm{RC}}(G,(x-1)(y-1),y-1).$$

1.5 Weighted homomorphisms into a matrix

Given a graph $G=(V,E)$, a $q\times q$ symmetric matrix N and a $\underline{\nu}\in\mathbb{R}^q$ we can consider

$$Z_G(N,\underline{\nu}) = \sum_{\varphi:V\to[q]} \prod_{(u,v)\in E} N_{\varphi(u),\varphi(v)} \cdot \prod_{u\in V} \nu_{\varphi(u)}.$$

Clearly, this generalises both the Ising model and the Potts model. In case of the Ising model, $q=2$ and $N=\begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$ and $\underline{\nu}=\begin{pmatrix} e^{B} \\ e^{-B} \end{pmatrix}$, while in the case of the Potts model we need to consider the matrix whose diagonal elements are $e^{\beta}=1+w$, and the off-diagonal elements are 1.

2. Edge activities

2.1 Introduction

In this chapter we introduce an alternative description of the Tutte polynomial. For Tutte it was a definition, for us it will be a theorem.

Theorem 2.1.1 (Tutte [48]). *Let G be a connected graph with m edges. Label the edges with $1, 2, \dots, m$ arbitrarily. In case of a spanning tree T of G, let us call an edge $e \in E(T)$ internally active if e has the largest label among the edges in the cut determined by T and e by removing e from T. Let us call an edge $e \notin E(T)$ externally active if e has the largest label among the edges in the cycle determined by T and e by adding e to T. Let $\mathrm{ia}(T)$ and $\mathrm{ea}(T)$ be the number of internally and externally active edges, respectively. Then*

$$T_G(x,y) = \sum_{T \in \mathcal{T}(G)} x^{\mathrm{ia}(T)} y^{\mathrm{ea}(T)},$$

where the summation goes for all spanning trees of G.

Theorem 2.1.1 was originally a definition for the Tutte polynomial [48]. This characterization of the Tutte polynomial immediately shows that the coefficients of the Tutte polynomial are non-negative. In this theorem, it is important that we consider the same labeling of the edges for all spanning trees. For those who have never seen this definition before, it might be very surprising that the Tutte polynomial is independent of the actual choice of the labeling.

Proof. For a spanning tree T let $\mathrm{IA}(T)$ be the set of internally active edges, and $\mathrm{EA}(T)$ be the set of externally active edges. Then we can write

$$\sum_{T \in \mathcal{T}(G)} x^{\mathrm{ia}(T)} y^{\mathrm{ea}(T)} = \sum_{T \in \mathcal{T}(G)} (1 + (x-1))^{\mathrm{ia}(T)} (1 + (y-1))^{\mathrm{ea}(T)}$$

$$= \sum_{T\in\mathcal{T}(G)} \sum_{S_1\subseteq \mathrm{IA}(T)} \sum_{S_2\subseteq \mathrm{EA}(T)} (x-1)^{|S_1|}(y-1)^{|S_2|}.$$

A very natural idea is to make a map $(T, S_1, S_2) \mapsto A \in 2^E$, and a natural candidate is $A = (T\setminus S_1)\cup S_2$. The question is whether we can get back (T, S_1, S_2) from A. This is again easy: think of the labels as weights, and in each component let us choose a minimum weight spanning tree, then contract the connected components of A and in the obtained graph choose a maximum weight spanning tree. Let S_1 be the edges of this latter graph, and let S_2 be those edges that were not in a minimum weight spanning tree of any connected component. Then $T = (A \setminus S_2) \cup S_1$ is a spanning tree. Observe that by the construction each edge of S_1 must be internally active with respect to T, and each edge of S_1 is externally active with respect to T. Since the edges of S_1 goes between different components of A we have $|S_1| = k(A) - k(E)$. The edges of S_2 goes inside the components of A we get that $|S_2| = |A| + k(A) - |V|$.

We show that it is a one-to-one map. Observe that an edge of S_2 cannot go between the connected components of $T \setminus S_1$, because that would mean that it has a bigger label than some internally active edge in its fundamental cycle, but it contradicts the definition of internal activity. This means that A and $T \setminus S_1$ has the same connected components. But it immediately determines S_1: only the maximum weight spanning tree in the contracted graph has the property that all of its edges are active. Similarly, only the minimum weight spanning tree in each connected component of A has the property that all other edges of A in the same connected component are externally active. This means that A uniquely determines T, S_1, S_2. □

2.2 Local basis exchange graph

Definition 2.2.1. The local basis exchange graph $H[T]$ of a graph $G = (V, E)$ with respect to a spanning tree T is defined as follows. The graph $H[T]$ is a bipartite graph whose vertices are the edges of G. One bipartite class consists of the edges of T, the other consists of the non-edges of T, and we connect a spanning tree edge e with a non-edge f if f is in the cut determined by e and T, equivalently, e is in the cycle determined by f and T.

Figure 1 depicts a graph G with a spanning tree T and the bipartite graph $H[T]$ obtained from T.

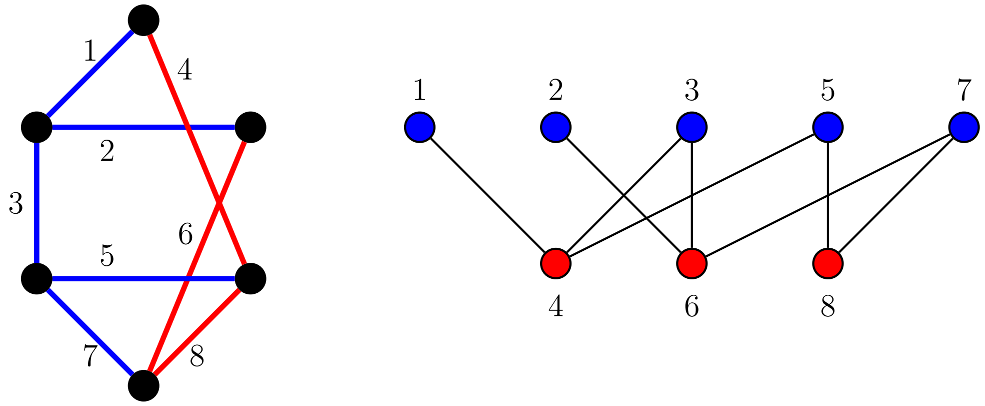


Figure 2.1: Example for a graph G and the local basis exchange graph $H[T]$ obtained from a spanning tree T.

2.3 Matroids

The Tutte polynomial naturally extends to matroids. Recall that a matroid M is a pair $(E, \mathcal{I})$ such that $\mathcal{I} \subseteq 2^E$, called the independent sets, satisfying the axioms (i) $\emptyset \in \mathcal{I}$, (ii) if $A' \subseteq A \in \mathcal{I}$, then $A' \in \mathcal{I}$, and (iii) if $A, B \in \mathcal{I}$ such that $|B| < |A|$, then there exists an $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$. Given a set $S \subseteq E$, the maximal independent subsets of S all have the same cardinality, and this cardinality is called the rank of the S, denoted by $r(S)$. The maximum size independent sets of M are called bases, and their set is denoted by $\mathcal{B}(M)$. The dual of a matroid M is the matroid M^* whose bases are $\{E \setminus B \mid B \in \mathcal{B}(M)\}$. For further details on matroids see for instance [44]

Given a graph $G = (V, E)$ the edge sets of the spanning forests of G form the independent sets of a matroid M_G called the cycle matroid of G. If G is connected, then the basis of M_G are the spanning trees of G. One can define the Tutte polynomial of a matroid as

$$T_M(x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)},$$

where $r(S)$ is the rank of a set $S \subseteq E$. When $M = M_G$, then $T_{M_G}(x, y) = T_G(x, y)$.

Observe that even the definition of local basis exchange works for general matroids and their basis.

3. Brylawski's identities

3.1 Introduction

In this chapter we study the so-called Brylawski's identities valid for the Tutte polynomial. We have seen that written as a usual two-variate polynomial $T_G(x,y) = \sum_{i,j} t_{ij}x^i y^j$, the coefficients t_{ij} encode the number of certain spanning trees, namely spanning trees with internal activity i and external activity j with respect to a fixed ordering of the edges. It is not hard to prove that $t_{00} = 0$ and $t_{10} = t_{01}$ if the graph G has at least 2 edges. In general, Brylawski [9] proved a collection of linear relations between the coefficients of the Tutte polynomial. Namely, he proved that if $e(G) \geq h+1$ for some $h \geq 0$, then

$$\sum_{i=0}^{h}\sum_{j=0}^{h-i}\binom{h-i}{j}(-1)^j t_{ij} = 0.$$

In particular, the third relation gives that if $e(G) \geq 3$, then $t_{20} - t_{11} + t_{02} = t_{10}$.

Example 3.1.1. The Tutte polynomial of the cube graph is the following.

	1	x	x^2	x^3	x^4	x^5	x^6	x^7
1		11	32	40	29	15	5	1
y	11	46	52	24				
y^2	25	39	16					
y^3	20	8						
y^4	7							
y^5	1							

We can see that $t_{10} = t_{01} = 11$ and $t_{20} - t_{11} + t_{01} = 32 - 46 + 25 = 11 = t_{10}$.

Note that one can extend Brylawski's identities to the Tutte polynomial of an arbitrary matroid M on a set E. In fact, we will only use the fact that there is a rank function $r(\cdot)$ on the subsets of E satisfying $r(S) \leq \min(r(E), |S|)$ for every set $S \subseteq E$. In this case, the Tutte polynomial of the system $M = (E, r)$ is defined as

$$T_M(x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)},$$

where $r(S)$ is the rank of a set $S \subseteq E$. The Tutte polynomial of a graph G simply corresponds to the graphical matroid M of the graph G. In [29], Gordon extended Brylawski's result for ranked sets: besides $r(S) \leq \min(r(E), |S|)$, he assumed the normalization $r(\emptyset) = 0$, and was able to extend Brylawski's identities for $h = |E|$.

Here we extend the work of Gordon and Brylawski for $h > |E|$, and also simplify the proof significantly. We only use the special form of the polynomial, namely that it behaves nicely along the hyperbola $(x-1)(y-1) = 1$. We do not use anything about matroids or rank functions. Our generalized Brylawski's identities are the following.

Theorem 3.1.2 (Generalized Brylawski's identities). *Let $M = (E, r)$, where E is a set, and $r : 2^E \to \mathbb{Z}_{\geq 0}$ is a rank function on the subsets of E, satisfying $r(S) \leq \min(r(E), |S|)$ for every set $S \subseteq E$. Let*

$$T_M(x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}$$

be the Tutte polynomial of the system $M = (E, r)$. Let m denote the size of E, and r the rank of E. By writing $T_M(x, y) = \sum_{i,j} t_{ij} x^i y^j$, the coefficients t_{ij} satisfy the following identities. For any integer $h \geq 0$, we have

$$\sum_{i=0}^{h} \sum_{j=0}^{h-i} \binom{h-i}{j} (-1)^j t_{ij} = (-1)^{m-r} \binom{h-r}{h-m},$$

with the convention that when $h < m$, the binomial coefficient $\binom{h-r}{h-m}$ is interpreted as 0.

In particular, for the graphical matroid of a graph G, we get the following.

Theorem 3.1.3 (Generalized Brylawski's identities for graphs). *Let G be any graph with n vertices, m edges and c connected components. Let $T_G(x, y) = \sum_{i,j} t_{ij} x^i y^j$ be the Tutte polynomial of the graph G. Then for any integer $h \geq 0$, we have*

$$\sum_{i=0}^{h} \sum_{j=0}^{h-i} \binom{h-i}{j} (-1)^j t_{ij} = (-1)^{m-n+c} \binom{h-n+c}{h-m},$$

with the convention that when $h < m$, the binomial coefficient $\binom{h-n+c}{h-m}$ is interpreted as 0.

3.2 Proof of Theorem 3.1.2

This entire section is devoted to the proof of Theorem 3.1.2.

By definition,

$$T_M(x,y) = \sum_{S\subseteq E} (x-1)^{r-r(S)}(y-1)^{|S|-r(S)}.$$

Let us introduce a new variable z, and plug in $x = \frac{z}{z-1}$ and $y = z$. Then

$$T_M\left(\frac{z}{z-1}, z\right) = \sum_{S\subseteq E} (z-1)^{|S|-r} = (z-1)^{-r} z^m = \frac{z^m}{(z-1)^r}.$$

Since $T_M(x,y) = \sum_{i,j} t_{ij} x^i y^j$, we have

$$T_M\left(\frac{z}{z-1}, z\right) = \sum_{i,j} t_{ij} \left(\frac{z}{z-1}\right)^i z^j = \frac{z^m}{(z-1)^r}.$$

Hence

$$\sum_{i,j} t_{i,j} z^{i+j} (z-1)^{r-i} = z^m.$$

Note that if $i > r$, then $t_{ij} = 0$ as $r(S) \geq 0$ for every set S. Hence, both sides are polynomials of z, so we can compare the coefficients of z^k.

$$\sum_{i,j} t_{i,j} (-1)^{r-k+j} \binom{r-i}{k-(i+j)} = \delta_{k,m},$$

where $\delta_{k,m}$ is 1 if $k = m$, and 0 otherwise. This is not yet exactly Brylawski's identity, but taking appropriate linear combinations of these equations yields Brylawski's identities. Let

$$C_{h,k} = (-1)^k \binom{h-r}{h-k}.$$

Then

$$\sum_{k=0}^{h} C_{h,k} \left(\sum_{i,j} t_{i,j} (-1)^{r-k+j} \binom{r-i}{k-(i+j)} \right) = C_{h,m}.$$

Let S_h be the left hand side. Note that

$$\begin{aligned} S_h &= \sum_{k=0}^{h} C_{h,k} \left(\sum_{i,j} t_{i,j} (-1)^{r-k+j} \binom{r-i}{k-(i+j)} \right) \\ &= \sum_{k=0}^{h} (-1)^k \binom{h-r}{h-k} \left(\sum_{i,j} t_{i,j} (-1)^{r-k+j} \binom{r-i}{k-(i+j)} \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} t_{i,j}(-1)^{r+j} \left(\sum_{k=0}^{h} \binom{h-r}{h-k} \binom{r-i}{k-(i+j)} \right) \\
&= \sum_{i,j} t_{i,j}(-1)^{r+j} \binom{h-i}{h-(i+j)} \\
&= \sum_{i,j} \binom{h-i}{j} t_{i,j}(-1)^{r+j}.
\end{aligned}
$$

Then

$$
\sum_{i,j} \binom{h-i}{j} t_{i,j}(-1)^{j} = (-1)^{m-r} \binom{h-r}{h-m}.
$$

Remark 3.2.1. Once one conjectures Theorem 3.1.3, then it can be proved by the deletion-contraction identities via simple induction on h. The more general Theorem 3.1.2 can be proved by certain recursions too, as it was shown by Gordon [29], but seems to be considerably more work than the proof presented above.

Remark 3.2.2. One key application of the Brylawski's identities is the characterization when $T_G(x, y)$ can be factorized: it is irreducible if and only if G is 2-connected.

4. The Merino–Welsh conjecture is false for matroids

4.1 Introduction

For a connected graph G, let $\tau(G)$, $a(G)$ and $a^*(G)$ denote the number of spanning trees, the number of acyclic orientations and the number of strongly connected orientations, respectively. Merino and Welsh [42] conjectured that if G is a connected graph without loops and bridges, then

$$\max(a(G), a^*(G)) \geq \tau(G).$$

We have seen that $a(G), a^*(G)$, and $\tau(G)$ are all evaluations of the Tutte polynomial, namely, $T_G(2,0) = a(G)$, $T_G(0,2) = a^*(G)$, and $T_G(1,1) = \tau(G)$.

Conde and Merino [20] also suggested the following "additive" and "multiplicative" versions of the conjecture:

$$T_G(2,0) + T_G(0,2) \geq 2T_G(1,1),$$

and

$$T_G(2,0)T_G(0,2) \geq T_G(1,1)^2.$$

It is easy to see that the multiplicative version implies the additive version which in turn implies the maximum version.

The Merino–Welsh conjecture and its variants triggered considerable attention. Thomassen [47] proved that the conjecture is true if the graph G is sufficiently sparse or sufficiently dense. Lin [35] proved it for 3-connected graphs satisfying certain degree conditions. Noble and Royle [43] proved the multiplicative version for series-parallel graphs.

As we have seen the Tutte polynomial naturally extends to matroids with the formula

$$T_M(x,y)=\sum_{S\subseteq E}(x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)},$$

where $r(S)$ is the rank of a set $S\subseteq E$. A loop in a matroid M is an element $x\in E$ such that $r(\{x\})=0$, that is, $\{x\}\notin\mathcal{I}$, and a coloop is an element that is a loop in the dual M^* of the matroid M. Equivalently, a coloop is an element that is in every base of M. For a cycle matroid M_G loops correspond to loop edges, coloops correspond to bridges in the graph G.

Hence it was suggested that the inequalities

$$\max(T_M(2,0),T_M(0,2)\geq T_M(1,1),$$

$$T_M(2,0)+T_M(0,2)\geq 2T_M(1,1),$$

$$T_M(2,0)T_M(0,2)\geq T_M(1,1)^2$$

may hold true for all matroids M without loops and coloops. (These versions appear explicitly in [26], but were treated much earlier without explicitly calling them conjectures.) Note that for general matroids, all these versions are equivalent in the following sense: if one of them is true for all matroids, then the others are also true for all matroids. Applying the maximum version to $M\cup M^*$ with M^* being the dual of M leads to the multiplicative version of the conjecture.

Knauer, Martínez-Sandoval, and Ramírez Alfonsín [33] proved that the class of lattice path matroids satisfy the multiplicative version. Ibañez, Merino and Rodríguez [40] proved the maximum version for some families of graphs and matroids. Chávez-Lomelí, Merino, Noble and Ramírez-Ibáñez [17] proved the additive version for paving matroids without coloops. In fact, they showed that the polynomial $T_M(x,2-x)$ is convex on the interval $[0,2]$ for these matroids. Recently, Ferroni and Schröter [26] proved the multiplicative version of the conjecture for split matroids. Kung [34] proved the additive version for some special matroids based on their size and rank. Jackson [31] proved that

$$T_M(3,0)T_M(0,3)\geq T_M(1,1)^2$$

for matroids without loops and coloops. He phrased it for graphs but he also noted that his proof extends to matroids.

The aim of this short note is to give a counter-example for these inequalities for general matroids.

Theorem 4.1.1. *There are infinitely many matroids M without loops and coloops for which*

$$T_M(2,0)T_M(0,2) < T_M(1,1)^2.$$

In fact, we show the following slightly stronger result. Let x_0 be the largest root of the polynomial $x^3 - 9(x-1)$. We have $x_0 \approx 2.22668...$

Theorem 4.1.2. *Let $0 \le x < x_0$, then there are infinitely many matroids M without loops and coloops for which*

$$T_M(x,0)T_M(0,x) < T_M(1,1)^2.$$

It is interesting to compare this result with the above inequality of Jackson. In the paper [5], the authors show that 3 can be improved to 2.9243.

4.2 Counter-examples*

The counter-example for the multiplicative version of the Merino–Welsh conjecture is surprisingly simple. Let $U_{n,r}$ be the uniform matroid on n elements with rank r. Let $U_{n,r}^{(2)}$ be the 2-thickening of $U_{n,r}$, that is, we replace each element of $U_{n,r}$ with 2 parallel elements. We will show that if $x < x_0$, then $M_n = U^{(2)}_{n,\frac{2}{3}n}$ satisfies the theorem for large enough n if n is divisible by 3, hence concluding Theorems 4.1.2 and 4.1.1.

The computation of the Tutte polynomial of $U_{n,r}^{(2)}$ relies on two well-known lemmas.

Lemma 4.2.1 (Formula (2.24) in [41])**.** *The Tutte polynomial of the matroid $U_{n,r}$ is the following:*

$$T_{U_{n,r}}(x,y) = \sum_{i=1}^{r} \binom{n-i-1}{n-r-1} x^i + \sum_{j=1}^{n-r} \binom{n-j-1}{r-1} y^j$$

if $0 < r < n$, and $T_{U_{n,n}}(x,y) = x^n$ and $T_{U_{n,0}}(x,y) = y^n$.

Lemma 4.2.2 (Jaeger, Vertigan and Welsh [32])**.** *Let M be a matroid, and let $M^{(k)}$ be its k-thickening, that is, we replace each element of M with k parallel elements. Then*

$$T_{M^{(k)}}(x,y) = (y^{k-1} + y^{k-2} + \cdots + 1)^{r(M)} T_M\left(\frac{y^{k-1} + y^{k-2} + \cdots + y + x}{y^{k-1} + y^{k-2} + \cdots + y + 1}, y^k\right).$$

In particular, we have

$$T_{M^{(2)}}(x,0) = T_M(x,0),$$
$$T_{M^{(2)}}(0,x) = (x+1)^{r(M)} T_M\left(\frac{x}{x+1}, x^2\right),$$

and

$$T_{M^{(2)}}(1,1) = 2^{r(M)} T_M(1,1).$$

Clearly, these expressions together with the exact formula for $T_{U_{n,r}}(x,y)$ make the computation of $T_{U^{(2)}_{n,r}}(x,y)$ very fast for specific values of n, r, x, y.

To prove Theorem 4.1.2, our next goal is to understand the exponential growth of $T_{U_{n,r}}(x,0)$.

Lemma 4.2.3. *Let $r = n\alpha$ and $x > 1$, then*

$$T_{U_{n,r}}(x,0) = \begin{cases} f(n)\exp(nH(\alpha)) & \text{if } x < \frac{1}{\alpha}, \\ f(n)(x(x-1)^{\alpha-1})^n & \text{if } x \geq \frac{1}{\alpha}, \end{cases}$$

where $n^K > f(n) > n^{-K}$ for some fixed K, and $H(\alpha) = -\alpha\ln(\alpha) - (1-\alpha)\ln(1-\alpha)$.

Proof. We can determine the dominating term of $T_{U_{n,r}}(x,0)$ by comparing two neighboring terms:

$$\binom{n-i-1}{n-r-1}x^i \geq \binom{n-i-2}{n-r-1}x^{i+1} \quad \text{if and only if} \quad \frac{n-i-1}{r-i} \geq x.$$

Hence, $\binom{n-i-1}{n-r-1}x^i$ is maximized at $\left\lceil \frac{xr-(n-1)}{x-1} \right\rceil$. If the right-hand side is negative, then the dominating term is at $i = 1$ and $\binom{n-2}{n-r-1} \sim \binom{n}{n-r} \sim \exp(nH(\alpha))$, where $\sim$ means the estimation is valid up to some n^K. When $x = \frac{1}{\alpha}$, then $\exp(H(\alpha)) = x(x-1)^{\alpha-1}$, so we can assume that $x \geq \frac{1}{\alpha}$ in the rest of the proof since then on the whole interval $\left(1, \frac{1}{\alpha}\right)$ we have $T_{U_{n,r}}(x,0) \sim \exp(nH(\alpha))$.

For the sake of simplicity, we carry out the estimation of the dominating term at

$$i = \frac{xr-n}{x-1} = \frac{x\alpha-1}{x-1}n$$

and we drop the integer part. All these changes affect our computation up to a term n^{-K}. In the forthcoming computation, we also estimate $m! \sim \left(\frac{m}{e}\right)^m$ as the terms $\sqrt{2\pi m}(1+o(1))$ can be integrated into $f(n)$:

$$\binom{n-i-1}{n-r-1}x^i \sim \binom{n-i}{n-r}x^i$$

$$
\begin{aligned}
&\sim \frac{\left(\frac{n-i}{e}\right)^{n-i}}{\left(\frac{n-r}{e}\right)^{n-r}\left(\frac{r-i}{e}\right)^{r-i}} x^i \\
&= \frac{\left(n\left(1-\frac{x\alpha-1}{x-1}\right)\right)^{n\left(1-\frac{x\alpha-1}{x-1}\right)}}{(n(1-\alpha))^{n(1-\alpha)}\left(n\left(\alpha-\frac{x\alpha-1}{x-1}\right)\right)^{n\left(\alpha-\frac{x\alpha-1}{x-1}\right)}} x^i \\
&= \left(\frac{\left(\frac{x(1-\alpha)}{x-1}\right)^{\frac{x(1-\alpha)}{x-1}}}{(1-\alpha)^{1-\alpha}\left(\frac{1-\alpha}{x-1}\right)^{\frac{1-\alpha}{x-1}}}\right)^n x^i \\
&= \left(x^{\frac{x(1-\alpha)}{x-1}}(x-1)^{\alpha-1}\right)^n x^i \\
&= \left(x^{\frac{x(1-\alpha)}{x-1}}(x-1)^{\alpha-1}\right)^n x^{\frac{x\alpha-1}{x-1}n} \\
&= (x(x-1)^{\alpha-1})^n,
\end{aligned}
$$

and the result follows.

□

Lemma 4.2.4. *Let $r = \alpha n$ and assume that $x \geq \frac{1}{\alpha}$ and $x^2 \geq \frac{1}{1-\alpha}$. Then for the matroid $M = U_{n,r}^{(2)}$, we have*

$$
\frac{T_M(1,1)^2}{T_M(x,0)T_M(0,x)} = g(n)\left(\frac{2^{2\alpha}}{\alpha^{2\alpha}(1-\alpha)^{2(1-\alpha)}} \cdot \frac{x-1}{x^3}\right)^n,
$$

where $n^K > g(n) > n^{-K}$ for some fixed K.

Proof. We have

$$
T_M(1,1) = 2^r T_{U_{n,r}}(1,1)) = 2^r \binom{n}{r} \sim \left(\frac{2^\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}}\right)^n.
$$

Furthermore,

$$
T_M(x,0) = T_{U_{n,r}}(x,0) \sim (x(x-1)^{\alpha-1})^n
$$

as $x \geq \frac{1}{\alpha}$. Finally,

$$
T_M(0,x) = (x+1)^r T_{U_{n,r}}\left(\frac{x}{x+1}, x^2\right) = (x+1)^r T_{U_{n,r}}\left(\frac{x}{x+1}, 0\right) + (x+1)^r T_{U_{n,r}}\left(0, x^2\right).
$$

Here, the second term will dominate the first one as $T_{U_{n,r}}\left(\frac{x}{x+1}, 0\right) < T_{U_{n,r}}(1,1) \sim \exp(nH(\alpha))$, while

$$
T_{U_{n,r}}\left(0, x^2\right) = T_{U_{n,n-r}}(x^2, 0) \sim (x^2(x^2-1)^{(1-\alpha)-1})^n
$$

as $x^2 \geq \frac{1}{1-\alpha}$. Putting everything together, we get that

$$\frac{T_M(1,1)^2}{T_M(x,0)T_M(0,x)} \sim \left(\frac{2^{2\alpha}}{\alpha^{2\alpha}(1-\alpha)^{2(1-\alpha)}}\right)^n (x(x-1)^{\alpha-1}(x+1)^{\alpha}x^2(x^2-1)^{-\alpha})^{-n}$$
$$\sim \left(\frac{2^{2\alpha}}{\alpha^{2\alpha}(1-\alpha)^{2(1-\alpha)}} \cdot \frac{x-1}{x^3}\right)^n.$$

□

Proof of Theorem 4.1.2. The maximum of the function $\frac{2^{2\alpha}}{\alpha^{2\alpha}(1-\alpha)^{2(1-\alpha)}}$ is at $\alpha = \frac{2}{3}$, where it takes value 9. We can assume by monotonicity that $2 \leq x < x_0$. Then $x \geq \frac{1}{\alpha} = \frac{3}{2}$ and $x^2 \geq \frac{1}{1-\alpha} = 3$, whence for $M = U^{(2)}_{n,\frac{2}{3}n}$,we get that

$$\frac{T_M(1,1)^2}{T_M(x,0)T_M(0,x)} \geq n^{-K}\left(\frac{9(x-1)}{x^3}\right)^n > 1$$

for large enough n as $\frac{9(x-1)}{x^3} > 1$. □

Remark 4.2.5. The matroid with the smallest number of elements that we are aware of being a counter-example to the multiplicative version of the Merino–Welsh conjecture is $M = U^{(2)}_{33,22}$ with 66 elements. For this matroid, we have $T_M(2,0) = 8374746166$, $T_M(0,2) = 64127582356390782814$, $T_M(1,1) = 811751838842880$, and

$$\frac{T_M(2,0)T_M(0,2)}{T_M(1,1)^2} \approx 0.815...$$

5. Correlation Inequalities

5.1 Introduction

Once we have such a measure we can ask that certain events Q_1, Q_2 are correlated or not, i. e., what the relation of $\mathbb{P}_\mu(Q_1 \cap Q_2)$ and $\mathbb{P}_\mu(Q_1)\mathbb{P}_\mu(Q_2)$ is. These are natural questions from a probabilistic point of view and as we will see these questions are also relevant for various other problems. For instance, knowing a certain correlation inequality can predict what kind of extremal graph theoretic results might hold.

5.2 Positive correlation

Definition 5.2.1. For $\underline{x}, \underline{y} \in \{0,1\}^n$ let $\underline{x} \vee \underline{y}$ be the vector for which $(\underline{x} \vee \underline{y})_i = \max(x_i, y_i)$, and let $\underline{x} \wedge \underline{y}$ be the vector for which $(\underline{x} \wedge \underline{y})_i = \min(x_i, y_i)$.

Theorem 5.2.2 (Ahlswede and Daykin [3])**.** *Let $f_1, f_2, f_3, f_4 : \{0,1\}^n \to \mathbb{R}_+$ satisfying the inequality*

$$f_1(\underline{x})f_2(\underline{y}) \leq f_3(\underline{x} \vee \underline{y})f_4(\underline{x} \wedge \underline{y})$$

for all $\underline{x}, \underline{y} \in \{0,1\}^n$. Let

$$F_i = \sum_{\underline{x} \in \{0,1\}^n} f_i(\underline{x})$$

for $i = 1, 2, 3, 4$. Then

$$F_1 \cdot F_2 \leq F_3 \cdot F_4.$$

Proof. We prove the statement by induction on n. For $n = 1$ the condition of the theorem gives that

$$f_1(0)f_2(0) \leq f_3(0)f_4(0).$$

$$f_1(0)f_2(1) \leq f_3(1)f_4(0).$$

$$f_1(1)f_2(0) \leq f_3(1)f_4(0).$$

$$f_1(1)f_2(1) \le f_3(1)f_4(1).$$

We need to prove that

$$(f_1(0)+f_1(1))(f_2(0)+f_2(1)) \le (f_3(0)+f_3(1))(f_4(0)+f_4(1)).$$

If $f_3(1)=0$ or $f_4(0)=0$ then $f_3(1)f_4(0) \le f_3(0)f_4(1)$ and the claim is trivially true:

$$(f_1(0)+f_1(1))(f_2(0)+f_2(1)) \le f_3(0)f_4(0)+2f_3(1)f_4(0)+f_3(1)f_4(1) \le (f_3(0)+f_3(1))(f_4(0)+f_4(1)).$$

So we can assume that $f_3(1)\ne 0$ and $f_4(0)\ne 0$. Then

$$(f_3(0)+f_3(1))(f_4(0)+f_4(1)) \ge \left(\frac{f_1(0)f_2(0)}{f_4(0)}+f_3(1)\right)\left(f_4(0)+\frac{f_1(1)f_2(1)}{f_3(1)}\right).$$

So it would be enough to prove that

$$\left(\frac{f_1(0)f_2(0)}{f_4(0)}+f_3(1)\right)\left(f_4(0)+\frac{f_1(1)f_2(1)}{f_3(1)}\right) \ge (f_1(0)+f_1(1))(f_2(0)+f_2(1)).$$

This is equivalent with

$$(f_1(0)f_2(0)+f_3(1)f_4(0))(f_3(1)f_4(0)+f_1(1)f_2(1)) \ge f_3(1)f_4(0)(f_1(0)+f_1(1))(f_2(0)+f_2(1)).$$

This is in turn equivalent with

$$(f_3(1)f_4(0)-f_1(0)f_2(1))(f_3(1)f_4(0)-f_1(1)f_2(0)) \ge 0$$

which is true by the assumptions of the theorem. This proves the case $n=1$.

Now suppose that the claim is true till $n-1$ and we wish to prove it for n. Set $f_i'(\underline{x}) : \{0,1\}^{n-1} \to \mathbb{R}_+$ for $i=1,2,3,4$ as follows:

$$f_i'(\underline{x}) = f_i(\underline{x},0)+f_i(\underline{x},1).$$

First we show that f_i' satisfies the inequality

$$f_1'(\underline{x})f_2'(\underline{y}) \le f_3'(\underline{x}\vee\underline{y})f_4'(\underline{x}\wedge\underline{y})$$

for all $\underline{x},\underline{y}\in\{0,1\}^{n-1}$. This is of course true: for a fixed $\underline{x},\underline{y}\in\{0,1\}^{n-1}$ let us apply the case $n=1$ to the functions

$$g_1(u)=f_1(\underline{x},u) \quad g_2(u)=f_2(\underline{y},u) \quad g_3(u)=f_3(\underline{x}\vee\underline{y},u) \quad g_4(u)=f_4(\underline{x}\wedge\underline{y},u),$$

where $u \in \{0,1\}$. Then the functions g_i satisfy

$$g_1(u_1)g_2(u_2) \le g_3(u_1 \vee u_2)g_4(u_1 \wedge u_2)$$

for all $u_1, u_2 \in \{0,1\}$ by the assumption on f. By the case $n = 1$ we know that

$$(g_1(0) + g_1(1))(g_2(0) + g_2(1)) \le (g_3(0) + g_3(1))(g_4(0) + g_4(1)).$$

In other words,

$$f_1'(\underline{x})f_2'(\underline{y}) \le f_3'(\underline{x} \vee \underline{y})f_4'(\underline{x} \wedge \underline{y})$$

for all $\underline{x}, \underline{y} \in \{0,1\}^{n-1}$. Then by induction we get that for $F_i' = \sum_{\underline{x} \in \{0,1\}^{n-1}} f_i'(\underline{x})$ we have

$$F_1' \cdot F_2' \le F_3' \cdot F_4'.$$

But of course $F_i' = F_i$ whence

$$F_1 \cdot F_2 \le F_3 \cdot F_4.$$

□

Theorem 5.2.3. *Let $f_1, f_2, f_3, f_4 : \{0,1\}^n \to \mathbb{R}_+$ satisfying the inequality*

$$f_1(\underline{x})f_2(\underline{y}) \le f_3(\underline{x} \vee \underline{y})f_4(\underline{x} \wedge \underline{y})$$

for all $\underline{x}, \underline{y} \in \{0,1\}^n$. Let $f_1', f_2', f_3', f_4'\{0,1\}^k \to \mathbb{R}_+$ be defined by

$$f_i'(\underline{x}) = \sum_{\underline{u} \in \{0,1\}^{n-k}} f_i(\underline{x}, \underline{u}).$$

Then for all $\underline{x}, \underline{y} \in \{0,1\}^k$ we have

$$f_1'(\underline{x})f_2'(\underline{y}) \le f_3'(\underline{x} \vee \underline{y})f_4'(\underline{x} \wedge \underline{y})$$

Proof. This immediately follows from Theorem 5.2.2. For fixed $\underline{x}, \underline{y} \in \{0,1\}^k$ define $g_1, g_2, g_3, g_4 : \{0,1\}^{n-k} \to \mathbb{R}_+$

$$g_1(\underline{u}) = f_1(\underline{x}, \underline{u}), \quad g_2(\underline{u}) = f_2(\underline{y}, \underline{u}), \quad g_3(\underline{u}) = f_3(\underline{x} \vee \underline{y}, \underline{u}), \quad g_4(\underline{u}) = f_4(\underline{x} \wedge \underline{y}, \underline{u}).$$

Then for any $\underline{u}, \underline{v} \in \{0,1\}^{n-k}$ we have

$$g_1(\underline{u})g_2(\underline{v}) \le g_3(\underline{u} \vee \underline{v})g_4(\underline{u} \wedge \underline{v})$$

by the assumption on the functions f_1, f_2, f_3, f_4. Then for

$$f_i'(\underline{x}) = G_i = \sum_{\underline{u}\in\{0,1\}^{n-k}} g_i(\underline{u}) = \sum_{\underline{u}\in\{0,1\}^{n-k}} f_i(\underline{x},\underline{u})$$

we have

$$f_1'(\underline{x})f_2'(\underline{y}) = G_1G_2 \leq G_3G_4 = f_3'(\underline{x}\vee\underline{y})f_4'(\underline{x}\wedge\underline{y}).$$

□

Definition 5.2.4. For $\underline{x},\underline{y} \in \{0,1\}^n$ we say that $\underline{x} \geq \underline{y}$ if for all $i \in [n]$ we have $x_i \geq y_i$.

A function $f : \{0,1\}^n \to \mathbb{R}^+$ is monotone increasing if $f(\underline{x}) \geq f(\underline{y})$ for all $\underline{x} \geq \underline{y}$ and it is monotone decreasing if $f(\underline{x}) \leq f(\underline{y})$ for all $\underline{x} \geq \underline{y}$.

In general, for a poset (or lattice) L a function $f : L \to \mathbb{R}^+$ is monotone increasing if $f(x) \geq f(x)$ for all $x \geq_L y$ and it is monotone decreasing if $f(x) \leq f(y)$ for all $x \geq_L y$.

Theorem 5.2.5. *A function $\mu : \{0,1\}^n \to \mathbb{R}^+$ is log-supermodular if*

$$\mu(\underline{x})\mu(\underline{y}) \leq \mu(\underline{x}\vee\underline{y})\mu(\underline{x}\wedge\underline{y})$$

for all $\underline{x},\underline{y} \in \{0,1\}^n$. Then for a log-supermodular $\mu : \{0,1\}^n \to \mathbb{R}^+$ and monotone increasing (decreasing) functions $f, g : \{0,1\}^n \to \mathbb{R}^+$ we have

$$\left(\sum_{\underline{x}\in\{0,1\}^n}\mu(\underline{x})f(\underline{x})\right)\left(\sum_{x\in\{0,1\}^n}\mu(\underline{x})g(\underline{x})\right) \leq \left(\sum_{\underline{x}\in\{0,1\}^n}\mu(\underline{x})f(\underline{x})g(\underline{x})\right)\left(\sum_{\underline{x}\in\{0,1\}^n}\mu(\underline{x})\right).$$

Furthermore, if $f : \{0,1\}^n \to \mathbb{R}^+$ is monotone increasing and $g : \{0,1\}^n \to \mathbb{R}^+$ is monotone decreasing then

$$\left(\sum_{\underline{x}\in\{0,1\}^n}\mu(\underline{x})f(\underline{x})\right)\left(\sum_{\underline{x}\in\{0,1\}^n}\mu(\underline{x})g(\underline{x})\right) \geq \left(\sum_{\underline{x}\in\{0,1\}^n}\mu(\underline{x})f(\underline{x})g(\underline{x})\right)\left(\sum_{\underline{x}\in\{0,1\}^n}\mu(\underline{x})\right).$$

Proof. First suppose that both f and g are monotone increasing. Let us apply Theorem 5.2.2 for the following theorems:

$$f_1(\underline{x}) = \mu(\underline{x})f(\underline{x}),\quad f_2(\underline{x}) = \mu(\underline{x})g(\underline{x}),\quad f_3(\underline{x}) = \mu(\underline{x})f(\underline{x})g(\underline{x}),\quad f_4(\underline{x}) = \mu(\underline{x}).$$

We need to check that

$$f_1(\underline{x})f_2(\underline{y}) \leq f_3(\underline{x}\vee\underline{y})f_4(\underline{x}\wedge\underline{y})$$

for all $\underline{x}, \underline{y} \in \{0,1\}^n$. This is indeed true:

$$\begin{aligned} f_1(\underline{x})f_2(\underline{y}) &= \mu(\underline{x})f(\underline{x})\mu(\underline{y})g(\underline{y}) \\ &\leq \mu(\underline{x} \vee \underline{y})\mu(\underline{x} \wedge \underline{y})f(\underline{x})g(\underline{y}) \\ &\leq \mu(\underline{x} \vee \underline{y})\mu(\underline{x} \wedge \underline{y})f(\underline{x} \vee \underline{y})g(\underline{x} \vee \underline{y}) \\ &= f_3(\underline{x} \vee \underline{y})f_4(\underline{x} \wedge \underline{y}). \end{aligned}$$

In the first inequality we used the log-supermodularity of μ, and in the second inequality we used that both f and g are monotone increasing. Then by Theorem 5.2.2 we have $F_1 \cdot F_2 \leq F_3 \leq F_4$, i. e.,

$$\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})\right)\left(\sum_{x\in\{0,1\}^n} \mu(\underline{x})g(\underline{x})\right) \leq \left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})g(\underline{x})\right)\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})\right).$$

If f and g are both monotone decreasing then set

$$f_1(\underline{x}) = \mu(\underline{x})f(\underline{x}), \quad f_2(\underline{x}) = \mu(\underline{x})g(\underline{x}), \quad f_3(\underline{x}) = \mu(\underline{x}), \quad f_4(\underline{x}) = \mu(\underline{x})f(\underline{x})g(\underline{x}).$$

Again we need to check that

$$f_1(\underline{x})f_2(\underline{y}) \leq f_3(\underline{x} \vee \underline{y})f_4(\underline{x} \wedge \underline{y})$$

for all $\underline{x}, \underline{y} \in \{0,1\}^n$. This is indeed true:

$$\begin{aligned} f_1(\underline{x})f_2(\underline{y}) &= \mu(\underline{x})f(\underline{x})\mu(\underline{y})g(\underline{y}) \\ &\leq \mu(\underline{x} \vee \underline{y})\mu(\underline{x} \wedge \underline{y})f(\underline{x})g(\underline{y}) \\ &\leq \mu(\underline{x} \vee \underline{y})\mu(\underline{x} \wedge \underline{y})f(\underline{x} \wedge \underline{y})g(\underline{x} \wedge \underline{y}) \\ &= f_3(\underline{x} \vee \underline{y})f_4(\underline{x} \wedge \underline{y}). \end{aligned}$$

From this we can conclude again that

$$\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})\right)\left(\sum_{x\in\{0,1\}^n} \mu(\underline{x})g(\underline{x})\right) \leq \left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})g(\underline{x})\right)\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})\right).$$

If f is monotone increasing, and g is monotone decreasing then let $M = \max_{\underline{x}\in\{0,1\}^n} g(\underline{x})$, and consider the function $g'(\underline{x}) = M - g(\underline{x})$. Then $g'(\underline{x}) \geq 0$ and monotone increasing. Whence

$$\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})\right)\left(\sum_{x\in\{0,1\}^n} \mu(\underline{x})g'(\underline{x})\right) \leq \left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})g'(\underline{x})\right)\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})\right).$$

By writing the definition of $g(\underline{x}) = M - g'(\underline{x})$ into it we get that

$$\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})\right)\left(\sum_{x\in\{0,1\}^n} \mu(\underline{x})(M-g(\underline{x}))\right) \leq \left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})(M-g(\underline{x}))\right)\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})\right).$$

After subtracting $M(\sum \mu(\underline{x}))(\sum \mu(\underline{x})f(\underline{x}))$ and multiplying with -1 we get that

$$\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})\right)\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})g(\underline{x})\right) \geq \left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})f(\underline{x})g(\underline{x})\right)\left(\sum_{\underline{x}\in\{0,1\}^n} \mu(\underline{x})\right).$$

□

Theorem 5.2.6. *Let L be a distributive lattice. A function $\mu : L \to \mathbb{R}^+$ is log-supermodular if*

$$\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$$

for all $x, y \in L$. For a log-supermodular $\mu : L \to \mathbb{R}^+$ and monotone increasing (decreasing) functions $f, g : L \to \mathbb{R}^+$we have

$$\left(\sum_{x\in L} \mu(x)f(x)\right)\left(\sum_{x\in L} \mu(x)g(x)\right) \leq \left(\sum_{x\in L} \mu(x)f(x)g(x)\right)\left(\sum_{x\in L} \mu(x)\right).$$

Furthermore, if $f : L \to \mathbb{R}^+$ is monotone increasing and $g : L \to \mathbb{R}^+$ is monotone decreasing then

$$\left(\sum_{x\in L} \mu(x)f(x)\right)\left(\sum_{x\in L} \mu(x)g(x)\right) \geq \left(\sum_{x\in L} \mu(x)f(x)g(x)\right)\left(\sum_{x\in L} \mu(x)\right).$$

Proof. This theorem follows from Theorem 5.2.5 since every distributive lattice L is a sublattice of some $\{0,1\}^n$. So all we need to do is to define μ on $\{0,1\}^n \setminus L$ to be 0, and to extend f and g in a monotone increasing way. (This last step is only needed formally since $\mu(\underline{x})f(\underline{x}), \mu(\underline{x})g(\underline{x}), \mu(\underline{x})f(\underline{x})g(\underline{x})$ are all 0 anyway for $\underline{x} \in \{0,1\}^n \setminus L$.) The extended μ will remain log-supermodular since $\mu(x)\mu(y) \neq 0$ then $x, y \in L$ and then $x \vee y, x \wedge y \in L$ so $\mu(x)\mu(y) \leq \mu(x \vee y)\mu(x \wedge y)$, and if $\mu(x)\mu(y) = 0$ then the inequality holds true trivially. □

In the next few results we give examples of various log-supermodular measures.

Theorem 5.2.7. *Assume that the function $\mu : \{0,1\}^n \to \mathbb{R}_+$ is log-supermodular. Then the function $\mu' : \{0,1\}^k \to \mathbb{R}_+$ defined by*

$$\mu'(\underline{x}) = \sum_{\underline{u}\in\{0,1\}^{n-k}} \mu(\underline{x}, \underline{u})$$

is also log-supermodular.

Proof. This theorem is an immediate application of Theorem 5.2.3 applied to $f_1 = f_2 = f_3 = f_4 = \mu$.

□

Theorem 5.2.8. *For probabilities* $p_1, \ldots, p_n$ *let*

$$\mathbb{P}_p(A) = \prod_{i \in A} p_i \prod_{j \notin A} (1 - p_j).$$

Let $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ *be monotone increasing, and* $\mathcal{C}, \mathcal{D} \subseteq 2^{[n]}$ *be monotone decreasing set families. For a set family* $\mathcal{S}$ *set*

$$\mathbb{P}_p(\mathcal{S}) = \sum_{S \in \mathcal{S}} \mathbb{P}_p(S).$$

Then we have

$$\mathbb{P}_p(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_p(\mathcal{A}) \cdot \mathbb{P}_p(\mathcal{B}),$$
$$\mathbb{P}_p(\mathcal{C} \cap \mathcal{D}) \geq \mathbb{P}_p(\mathcal{C}) \cdot \mathbb{P}_p(\mathcal{D}),$$
$$\mathbb{P}_p(\mathcal{A} \cap \mathcal{C}) \leq \mathbb{P}_p(\mathcal{A}) \cdot \mathbb{P}_p(\mathcal{C}).$$

Proof. We can associate the characteristic vector $\underline{1}_A \in \{0,1\}^n$ with a set A. Let

$$\mu(\underline{x}) = \prod_{i=1}^{n} p_i^{x_i} (1 - p_i)^{1 - x_i}.$$

Then $\mathbb{P}_p(A) = \mu(\underline{1}_A)$. Then

$$\mu(\underline{x})\mu(\underline{y}) = \mu(\underline{x} \vee \underline{y})\mu(\underline{x} \wedge \underline{y})$$

or equivalently $\mathbb{P}_p(A)\mathbb{P}_p(B) = \mathbb{P}_p(A \cup B)\mathbb{P}_p(A \cap B)$. Furthermore, let f be the characteristic functions of the family of sets $\mathcal{A}$, i. e., $f(1_A) = 1$ if $A \in \mathcal{A}$ and 0 otherwise. Similarly, let g be the characteristic functions of the family of sets $\mathcal{B}$. Then f and g are monotone increasing functions. The inequality

$$\mathbb{P}_p(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_p(\mathcal{A}) \cdot \mathbb{P}_p(\mathcal{B})$$

is simply Theorem 5.2.5 applied to μ, f and g. The other parts of the theorem follows similarly.

□

6. Permutation Tutte polynomial

6.1 Introduction

The aim of this chapter is to introduce an auxiliary polynomial that helps studying the Tutte polynomial and has properties that make it interesting even on its own. We call this new polynomial the *permutation Tutte polynomial*. It is defined for every bipartite graph.

Definition 6.1.1. Let $H = (A, B, E)$ be a bipartite graph. Suppose that $V(H) = [m]$. For a permutation $\pi : [m] \to [m]$, we say that a vertex $i \in A$ is internally active if

$$\pi(i) > \max_{j \in N_H(i)} \pi(j).$$

Similarly, we say that vertex $j \in B$ is externally active if

$$\pi(j) > \max_{i \in N_H(j)} \pi(i).$$

Let $\mathrm{ia}(\pi)$ and $\mathrm{ea}(\pi)$ be the number of internally and externally active vertices in A and B, respectively. Let

$$\widetilde{T}_H(x, y) = \frac{1}{m!} \sum_{\pi \in S_m} x^{\mathrm{ia}(\pi)} y^{\mathrm{ea}(\pi)}.$$

We will call $\widetilde{T}_H(x, y)$ the permutation Tutte polynomial of H.

Example 6.1.2. Let $H = P_5$ be the path on 5 vertices where A consists of 3, B consists of 2 vertices, respectively. The permutation Tutte polynomial of it is the following.

	1	x	x^2	x^3
1		$\frac{1}{15}$	$\frac{4}{15}$	$\frac{2}{15}$
y	$\frac{1}{15}$	$\frac{1}{3}$		
y^2	$\frac{2}{15}$			

This also satisfies Brylawski's identities: for instance, we can see that $t_{10} = t_{01} = \frac{1}{15}$ and $t_{20} - t_{11} + t_{01} = \frac{2}{15} - \frac{1}{3} + \frac{4}{15} = \frac{1}{15} = t_{10}$. For bipartite graphs without isolated vertices it is also true that $t_{a,0} = t_{0,b}$, in our example $t_{30} = t_{02} = \frac{2}{15}$

Example 6.1.3. The permutation Tutte polynomial of the cube graph is the following.

	1	x	x^2	x^3	x^4
1		$\frac{3}{28}$	$\frac{5}{28}$	$\frac{1}{14}$	$\frac{1}{56}$
y	$\frac{3}{28}$	$\frac{1}{4}$			
y^2	$\frac{5}{28}$				
y^3	$\frac{1}{14}$				
y^4	$\frac{1}{56}$				

The motivation behind Definition 6.1.1 comes from the characterisation of the Tutte polynomial via edge activities. In order to explain the connection between $T_G(x, y)$ and $\widetilde{T}_H(x, y)$, recall the concept of the local basis exchange graph.

For a fixed labeling of the edges of G, we get a labeling of the vertices of $H[T]$, and the internally (externally) active edges of G correspond to internally (externally) active vertices of $H[T]$, so the two definitions of internal and external activity are compatible. Taking all permutations of the edge labels and averaging out will correspond to averaging out the constant $T_G(x, y)$ on the level of G, and will lead to the definition of $\widetilde{T}_{H[T]}(x, y)$. This gives the identity

$$T_G(x, y) = \sum_{T \in \mathcal{T}(G)} \widetilde{T}_{H[T]}(x, y),$$

where the summation goes for all spanning trees of G (see Lemma 6.3.1 for further details). This identity is the starting point of several proofs of our theorems concerning the Tutte polynomial.

Jackson [31] proved the inequality:

$$T_G(3, 0)T_G(0, 3) \geq T_G(1, 1)^2$$

for every graph G without loops and bridges. In this chapter we show that

$$\widetilde{T}_H(3, 0)\widetilde{T}_H(0, 3) \geq \widetilde{T}_H(1, 1)^2$$

holds true for every bipartite graph H without isolated vertices, and this inequality implies Jackson's inequality (see transfer lemma, Lemma 6.3.3). This proof is completely different from the original proof of Jackson's inequality. The proof uses the FKG-inequality, and relies on the fact that permutations on m elements can be generated by simply ordering m random numbers chosen uniformly from $[0,1]$. This idea is the heart of several inequalities for $\widetilde{T}_H(x,y)$, and this is the key advantage of $\widetilde{T}_H(x,y)$ over $T_G(x,y)$.

We have seen that the Merino–Welsh conjecture is not true for all matroids [4], implying that $\widetilde{T}_H(2,0)\widetilde{T}_H(0,2) \geq \widetilde{T}_H(1,1)^2$ is not true for all bipartite graphs without isolated vertices. Nevertheless, there are several graph classes for which $\widetilde{T}_H(2,0)\widetilde{T}_H(0,2) \geq \widetilde{T}_H(1,1)^2$ holds true, including complete bipartite graphs, regular bipartite graphs and trees. One can also improve on Jackson's inequality by showing that $\widetilde{T}_H(x,0)\widetilde{T}_H(0,x) \geq \widetilde{T}_H(1,1)^2$ for every bipartite graph without isolated vertices if $x \geq 2.9243$. By the transfer lemma, this implies that $T_G(x,0)T_G(0,x) \geq T_G(1,1)^2$ for every graph G without loops and bridges (and matroids without loops and coloops).

6.2 Basic recursions

In this section, we establish several basic recursions for the permutation Tutte polynomial that we will use subsequently. The following lemmas are trivial.

Lemma 6.2.1. *If H is the disjoint union of H_1 and H_2, then*

$$\widetilde{T}_H(x,y) = \widetilde{T}_{H_1}(x,y)\widetilde{T}_{H_2}(x,y).$$

In particular, if $v \in A$ is an isolated vertex, then

$$\widetilde{T}_H(x,y) = x\widetilde{T}_{H-v}(x,y).$$

Similarly, if $v \in B$ is an isolated vertex, then

$$\widetilde{T}_H(x,y) = y\widetilde{T}_{H-v}(x,y).$$

Lemma 6.2.2. *For a bipartite graph $H = (A,B,E)$, let $H' = (B,A,E)$ be the graph obtained by switching the two sides of H. Then*

$$\widetilde{T}_H(x,y) = \widetilde{T}_{H'}(y,x).$$

Lemma 6.2.3. *If H is a bipartite graph on m vertices that does not contain isolated vertices, then*

$$\widetilde{T}_H(x,y) = \frac{1}{m} \sum_{v \in V(H)} \widetilde{T}_{H-v}(x,y).$$

Proof. For $\pi \in S_m$, let $v(\pi)$ be the vertex of H such that $\pi(v(\pi)) = 1$. Let $\alpha(\pi)$ be the permutation of $V(H - v(\pi))$ where $\alpha(x) < \alpha(y)$ iff $\pi(x) < \pi(y)$. Then a vertex is internally (externally) active in α if and only if it is internally (externally) active in π, since $v(\pi)$ cannot be active as $v(\pi)$ is not isolated. Therefore $\mathrm{ia}_H(\pi) = \mathrm{ia}_{H-v(\pi)}(\alpha(\pi))$ and $\mathrm{ea}_H(\pi) = \mathrm{ea}_{H-v(\pi)}(\alpha(\pi))$. As π runs through S_m, we remove each vertex $v \in V(H)$ exactly $(m-1)!$ times and get each permutation α of $\mathrm{Sym}\left([m]\backslash\{v\}\right)$ exactly once, so

$$\begin{aligned}
\widetilde{T}_H(x,y) &= \frac{1}{m!} \sum_{\pi \in S_m} x^{\mathrm{ia}_H(\pi)} y^{\mathrm{ea}_H(\pi)} \\
&= \frac{1}{m!} \sum_{\pi \in S_m} x^{\mathrm{ia}_{H-v(\pi)}(\alpha(\pi))} y^{\mathrm{ea}_{H-v(\pi)}(\alpha(\pi))} \\
&= \frac{1}{m!} \sum_{v \in V(H)} \sum_{\alpha \in Sym([m]\backslash\{v\})} x^{\mathrm{ia}_{H-v}(\alpha)} y^{\mathrm{ea}_{H-v}(\alpha)} \\
&= \frac{1}{m} \sum_{v \in V(H)} \widetilde{T}_{H-v}(x,y).
\end{aligned}$$

□

6.3 Connection with the Tutte polynomial

In this section, we establish the main connection between the Tutte polynomial and the permutation Tutte polynomial. This connection will enable us to transfer linear identities and inequalities from the permutation Tutte polynomial to the Tutte polynomial.

Lemma 6.3.1. *Let G be a graph. For each spanning tree T of G, let $H[T]$ be the local basis exchange graph with respect to T. Then*

$$T_G(x,y) = \sum_{T \in \mathcal{T}(G)} \widetilde{T}_{H[T]}(x,y),$$

where the sum is over the set of spanning trees $\mathcal{T}(G)$ of G.

Proof. For a fixed spanning tree T and a permutation π of the edges, the internally and externally active edges correspond to the internally and externally active vertices of $H[T]$. Hence

$$T_G(x,y) = \sum_{T \in \mathcal{T}(G)} x^{\mathrm{ia}_{H[T]}(\pi)} y^{\mathrm{ea}_{H[T]}(\pi)}.$$

Now averaging it for all permutations $\pi \in S_m$ we get that

$$\begin{aligned} T_G(x,y) &= \frac{1}{m!} \sum_{\pi \in S_m} T_G(x,y) \\ &= \frac{1}{m!} \sum_{\pi \in S_m} \sum_{T \in \mathcal{T}(G)} x^{\mathrm{ia}_{H[T]}(\pi)} y^{\mathrm{ea}_{H[T]}(\pi)} \\ &= \sum_{T \in \mathcal{T}(G)} \frac{1}{m!} \sum_{\pi \in S_m} x^{\mathrm{ia}_{H[T]}(\pi)} y^{\mathrm{ea}_{H[T]}(\pi)} \\ &= \sum_{T \in \mathcal{T}(G)} \widetilde{T}_{H[T]}(x,y). \end{aligned}$$

□

Remark 6.3.2. The local basis exchange graph $H[T]$ has an isolated vertex if and only if G contains a bridge or a loop. Furthermore, $H[T]$ is connected if and only G is 2-connected.

The following lemma enables us to study Conde-Merino-Welsh type inequalities.

Lemma 6.3.3 (Transfer lemma)**.** *Let $x_0, x_1, x_2, y_0, y_1, y_2 \geq 0$. Suppose that for any bipartite graph H, we have*

$$\widetilde{T}_H(x_1,y_1)\widetilde{T}_H(x_2,y_2) \geq \widetilde{T}_H(x_0,y_0)^2.$$

Then for any graph G, we have

$$T_G(x_1,y_1)T_G(x_2,y_2) \geq T_G(x_0,y_0)^2.$$

More generally, if for $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n \geq 0$ and $\alpha_1, \dots, \alpha_n \geq 0$ satisfying $\sum_{k=1}^n \alpha_k = 1$, the inequality

$$\prod_{k=1}^n \widetilde{T}_H(x_k,y_k)^{\alpha_k} \geq \widetilde{T}_H(x_0,y_0)$$

holds true for every bipartite graph H, then for every graph G, we have

$$\prod_{k=1}^n T_G(x_k,y_k)^{\alpha_k} \geq T_G(x_0,y_0).$$

Proof. We have

$$\begin{aligned} T_G(x_1,y_1)T_G(x_2,y_2) &= \left(\sum_{T\in\mathcal{T}(G)} \widetilde{T}_{H[T]}(x_1,y_1)\right)\left(\sum_{T\in\mathcal{T}(G)} \widetilde{T}_{H[T]}(x_2,y_2)\right) \\ &\geq \left(\sum_{T\in\mathcal{T}(G)} \left(\widetilde{T}_{H[T]}(x_1,y_1)\widetilde{T}_{H[T]}(x_2,y_2)\right)^{1/2}\right)^2 \\ &\geq \left(\sum_{T\in\mathcal{T}(G)} \widetilde{T}_{H[T]}(x_0,y_0)\right)^2 \\ &= T_G(x_0,y_0)^2. \end{aligned}$$

The first and last equality are the applications Lemma 6.3.1. The first inequality is a Cauchy–Schwarz inequality applied to the numbers $\widetilde{T}_{H[T]}(x_1,y_1)^{1/2}, \widetilde{T}_{H[T]}(x_2,y_2)^{1/2}$ for $T\in\mathcal{T}(G)$. This is where we use that $x_1,x_2,x_3,y_1,y_2,y_3\geq 0$ to ensure that we can consider the square roots. The second inequality is simply the condition of the lemma.

The proof of the more general statement follows the same way, the only difference is that Cauchy–Schwarz inequality we have to use the following version of Hölder's inequality:

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{M} a_{kj}\right)^{\alpha_k} \geq \sum_{j=1}^{M}\prod_{k=1}^{n} a_{kj}^{\alpha_k}.$$

□

6.4 Applications of the FKG-inequality

In this section, we show the advantage of $\widetilde{T}_H(x,y)$ over $T_G(x,y)$ in proving Conde-Merino-Welsh-type inequalities.

Let us immediately give two inequalities as motivations.

Lemma 6.4.1. *Let H be an arbitrary bipartite graph. Suppose that $0\leq x\leq 1$ and $y\geq 1$ or $0\leq y\leq 1$ and $x\geq 1$. Then*

$$\widetilde{T}_H(x,y)\widetilde{T}_H(1,1) \geq \widetilde{T}_H(x,1)\widetilde{T}_H(1,y).$$

If both $x,y\geq 1$ or both $0\leq x,y\leq 1$, then

$$\widetilde{T}_H(x,y)\widetilde{T}_H(1,1) \leq \widetilde{T}_H(x,1)\widetilde{T}_H(1,y).$$

Note that $\widetilde{T}_H(1,1)=1$, so it appears in the lemma only for aesthetic reasons.

Lemma 6.4.2. *Let H be an arbitrary bipartite graph, and let d_i be the degree of a vertex i. Suppose that $0 \le x \le 1$ and $y \ge 1$ or $0 \le y \le 1$ and $x \ge 1$. Then*

$$\widetilde{T}_H(x,y) \ge \prod_{i\in A}\left(1+\frac{x-1}{d_i+1}\right)\cdot\prod_{j\in B}\left(1+\frac{y-1}{d_j+1}\right).$$

To prove Lemma 6.4.1 and 6.4.2, we need the following form of the FKG-inequality [27].

Lemma 6.4.3 (Fortuin, Kasteleyn, Ginibre [27])**.** *Suppose that μ is the uniform measure on $[0,1]^N$, and $X_1,\dots,X_t$ are non-negative monotone increasing functions in the sense that if $x_i \ge x_i'$ for $i=1,\dots,N$, then for $1 \le j \le t$ we have*

$$X_j(x_1,\dots,x_n) \ge X_j(x_1',\dots,x_n').$$

Then

$$\mathbb{E}_\mu\left[\prod_{j=1}^t X_j\right] \ge \prod_{j=1}^t \mathbb{E}_\mu[X_j].$$

Furthermore, if X is monotone increasing and Y is monotone decreasing, then

$$\mathbb{E}[XY] \le \mathbb{E}[X]\mathbb{E}[Y].$$

Remark 6.4.4. The above special case of the FKG-inequality might be non-standard, because it is mostly stated for finite distributive lattices, but $[0,1]^N$ can be approximated by the finite distributive lattices $\left\{0,\frac{1}{M},\frac{2}{M},\dots,1\right\}^N$, so the above version follows from the usual versions. It can also be proved by induction on n by repeatedly using the case $t=2, n=1$, which is just Chebyshev's inequality.

In what follows, we repeatedly use the same idea to express $\widetilde{T}_H(x,y)$. This is a crucial idea.

We can create a random ordering of the vertices of H as follows: for each vertex i we choose a uniform random number x_i from the interval $[0,1]$. The numbers x_i then determine an ordering of the edges. The probability that two numbers are equal is 0. For $j \in B$ we actually first generate a uniformly random number y_j from $[0,1]$ and let $y_j = 1-x_j$. This trick makes some argument more convenient.

For $i \in A$, let us introduce the random variable

$$X_i(x_i,\{y_j\}_{j\in B}) = \begin{cases} x & \text{if } \max_{j\in N_H(i)}(1-y_j) \le x_i, \\ 1 & \text{if } \max_{j\in N_H(i)}(1-y_j) > x_i. \end{cases}$$

and for $j \in B$, let

$$Y_j(\{x_i\}_{i\in A}, y_j) = \begin{cases} y & \text{if } \max_{i\in N_H(j)} x_i \le 1 - y_j, \\ 1 & \text{if } \max_{i\in N_H(j)} x_i \ge 1 - y_i. \end{cases}$$

Since we simply generated a uniform random ordering of the vertices, we get that

$$\widetilde{T}_H(x,y) = \mathbb{E}\left[\prod_{i\in A} X_i \cdot \prod_{j\in B} Y_j\right].$$

Now observe that if $x \ge 1$, then X_i is a monotone increasing function, and if $x \le 1$, then it is a monotone decreasing function of $(x_i)_{i\in A}, (y_j)_{j\in B}$. Indeed, if it was true that $\max_{j\in N_H(i)}(1-y_j) \le x_i$, then this inequality remains true if we increase x_i and y_j. So X_i is monotone increasing if $x \ge 1$, and decreasing if $0 \le x \le 1$. In case of $j \in B$, we get that Y_j is increasing if $0 \le y \le 1$ and decreasing if $y \ge 1$.

Now we are ready to prove Lemma 6.4.1 and 6.4.2.

Proof of Lemma 6.4.1. If $x \ge 1$ and $0 \le y \le 1$, then $\prod_{i\in A} X_i$ and $\prod_{j\in B} Y_j$ are both monotone increasing random variables. Hence

$$\widetilde{T}_H(x,y) = \mathbb{E}\left[\prod_{i\in A} X_i \cdot \prod_{j\in B} Y_j\right] \ge \mathbb{E}\left[\prod_{i\in A} X_i\right] \cdot \mathbb{E}\left[\prod_{j\in B} Y_j\right] = \widetilde{T}_H(x,1)\widetilde{T}_H(1,y).$$

The other inequalities follow the same way. □

Proof of Lemma 6.4.2. We have

$$\mathbb{E}[X_i] = \left(1 - \frac{1}{d_i+1}\right) + \frac{x}{d_i+1} = 1 + \frac{x-1}{d_i+1},$$

and

$$\mathbb{E}[Y_j] = \left(1 - \frac{1}{d_j+1}\right) + \frac{y}{d_j+1} = 1 + \frac{y-1}{d_j+1}.$$

Note that X_i and Y_j are monotone increasing functions in terms of the variables $\{x_i\}_{i\in A}$ and $\{y_j\}_{j\in B}$ if $x \ge 1$ and $0 \le y \le 1$, and they are monotone decreasing functions in terms of the variables $\{x_i\}_{i\in A}$ and $\{y_j\}_{j\in B}$ if $0 \le x \le 1$ and $y \ge 1$. Hence, by the FKG-inequality, we have

$$\widetilde{T}_H(x,y) = \mathbb{E}\left[\prod_{i\in A} X_i \cdot \prod_{j\in B} Y_j\right] \ge \prod_{i\in A} \mathbb{E}[X_i] \cdot \prod_{j\in B} \mathbb{E}[Y_j] = \prod_{i\in A}\left(1 + \frac{x-1}{d_i+1}\right) \cdot \prod_{j\in B}\left(1 + \frac{y-1}{d_j+1}\right).$$

□

An interesting application of the above inequalities is the following. (This inequality is particularly useful if one studies graphs with large girth, and a variant of this inequality was used in the paper [6].)

Theorem 6.4.5. *Suppose that a graph G has n vertices, m edges and the length of the shortest cycle is g. Then*

$$T_G(x,0) \geq T_G(x,1)\left(1-\frac{1}{g}\right)^{m-n+1}.$$

Proof. For any spanning tree T the local basis exchange graph $H = H[T]$ has minimum degree $g-1$ on the side of the non-spanning-tree edges. This means that if $x \geq 1$, we have

$$\widetilde{T}_H(x,0) \geq \widetilde{T}_H(x,1)\widetilde{T}_H(1,0) \geq \widetilde{T}_H(x,1)\prod_{j\in B}\left(1-\frac{1}{d_j+1}\right) \geq \widetilde{T}_H(x,1)\left(1-\frac{1}{g}\right)^{m-n+1}.$$

By summing this inequality for all spanning trees, we get that

$$T_G(x,0) \geq T_G(x,1)\left(1-\frac{1}{g}\right)^{m-n+1}.$$

□

Another application gives a fast proof of Jackson's theorem.

Theorem 6.4.6. *Let H be a bipartite graph with minimum degree $\delta \geq 1$. Then*

$$\widetilde{T}_H\left(2+\frac{1}{\delta},0\right)\widetilde{T}_H\left(0,2+\frac{1}{\delta}\right) \geq \widetilde{T}_H(1,1)^2.$$

In particular, we have

$$\widetilde{T}_H(3,0)\widetilde{T}_H(0,3) \geq \widetilde{T}_H(1,1)^2.$$

Let G be a graph without loops and bridges. Then

$$T_G(3,0)T_G(0,3) \geq T_G(1,1)^2.$$

Proof. Let $x = 2+\frac{1}{\delta}$. Let us use that $\widetilde{T}_H(1,1) = 1$,

$$\widetilde{T}_H(x,0) \geq \prod_{i\in A}\left(1+\frac{x-1}{d_i+1}\right)\cdot\prod_{j\in B}\left(1-\frac{1}{d_j+1}\right),$$

and

$$\widetilde{T}_H(0,x) \geq \prod_{i\in A}\left(1-\frac{1}{d_i+1}\right)\cdot\prod_{j\in B}\left(1+\frac{x-1}{d_j+1}\right).$$

So it is enough to prove that

$$\left(1+\frac{x-1}{d_v+1}\right)\left(1-\frac{1}{d_v+1}\right)\geq 1$$

if d_v is the degree of a vertex v. The inequality $(1+(x-1)\varepsilon)(1-\varepsilon)\geq 1$ is equivalent with $(x-2)\varepsilon \geq (x-1)\varepsilon^2$, that is, $\varepsilon \leq \frac{x-2}{x-1} = \frac{1}{\delta+1}$ which is satisfied since $d_v \geq \delta$ for all vertex $v \in V(H)$. The second inequality follows from the first one by simply taking $\delta = 1$. The third inequality follows from the second one by Lemma 6.3.3. □

Part II

Limits of Tutte polynomials

7. Limits of Tutte polynomials: general plan

In the second part of this course we aim to study the following problem. Given a "converging" graph sequence $(G_n)_n$ what is the limit

$$\lim_{n\to\infty} T_{G_n}(x,y)^{1/v(G_n)},$$

does it exist at all? To make this question rigorous we will later introduce the concept of the Benjamini–Schramm convergence.

Unfortunately, it turns out that our knowledge is very limited about this question. Even if we consider larger and larger grids we only know the limit value if $x = 1, y = 1$ or $(x-1)(y-1) \in \{1,2\}$. But at least in this case we know that the limit exists for all $x, y \geq 0$.

We will study another case when G_n are d-regular graphs converging to the infinite d-regular tree. This special case can be imagined as follows: we consider graphs that do not contain short cycles, in fact, as n grows, the length of the shortest cycle in G_n is longer and longer. In this special case we do not know that the limit exists for all $x, y \geq 0$, but it is widely believed so. Nevertheless we will prove the existence of the limit and compute it explicitly in two different regions. The first region is when $x \geq 1$ and $0 \leq y \leq 1$, the second region consists of the points (x, y) satisfying $(x-1)(y-1) \geq 2$ and $x, y > 1$. While the proofs in the two regions are different, the underlying ideas follow the same pattern. In both cases we first "approximate" the Tutte polynomial with another graph polynomial, then use the special properties of the approximating polynomial to deduce the existence of the limit and compute it precisely. The aforementioned special property is the special locus of zeros of the graph polynomial: in the first case they are real, in the second case they will be of unit length.

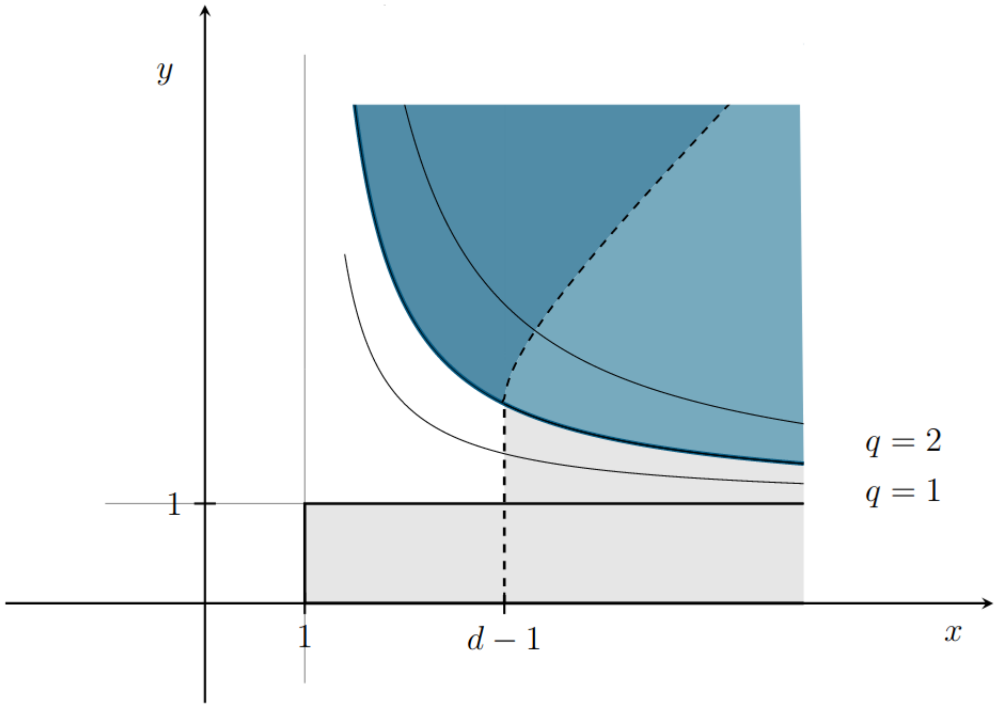


Figure 7.1: The two regions. Here $q = (x-1)(y-1)$ and the dashed line depicts the phase transition.

	Region 1	Region 2
Description	$x \geq 1$ and $0 \leq y \leq 1$	$(x-1)(y-1) \geq 2$ and $x, y > 1$
Approximating polynomial	Matching polynomial	Subgraph counting polynomial
Locus of zeros	Real line	Unit circle
Limit value	Theorem 1	Theorem 2

Below we give the exact theorems for the two regions.

Theorem 1. (Region 1) *Let $x \geq 1$ and $0 \leq y \leq 1$. Let $d \geq 2$, and let $(G_n)_n$ be a sequence of d-regular graphs such that $\lim_{n\to\infty} g(G_n) = \infty$. Then*

$$\lim_{n\to\infty} T_{G_n}(x,y)^{1/v(G_n)} = t_d(x,y),$$

where

$$t_d(x,y) = \begin{cases} (d-1)\left(\frac{(d-1)^2}{(d-1)^2-x}\right)^{d/2-1} & \textit{if } x \leq d-1 \textit{ and } \ 0 \leq y \leq 1, \\ x\left(1+\frac{1}{x-1}\right)^{d/2-1} & \textit{if } x > d-1 \textit{ and } \ 0 \leq y \leq 1. \end{cases}$$

If $(G_n)_n$ *is a sequence of random d-regular graphs, then the same statement holds true asymptotically almost surely. In fact, if* $L(G,g)$ *denotes the number of cycles of length at most* $g-1$ *in a graph* G*, then the same conclusion holds if for every fixed* g *we have* $\lim_{n\to\infty} \frac{L(G_n,g)}{v(G_n)} = 0$.

In case of region 2 it is better to use the parameterization $q = (x-1)(y-1)$ and $w = y-1$ and the partition function $Z_G(q,w) = \sum_{A\subseteq E} q^{k(A)} w^{|A|}$ instead of $T_G(x,y)$.

Theorem 2. (Region 2) *Let* $(G_n)_n$ *be a sequence of d-regular graphs such that* $\lim_{n\to\infty} g(G_n) = \infty$*. Then the limit*

$$\lim_{n\to\infty} Z_{G_n}(q,w)^{1/v(G_n)} = \Phi_{d,q,w}$$

exists for $q \geq 2$ *and* $w \geq 0$*. The quantity* $\Phi_{d,q,w}$ *can be computed as follows. Let*

$$\Phi_{d,q,w}(t) := \left(\sqrt{1+\frac{w}{q}}\cos(t) + \sqrt{\frac{(q-1)w}{q}}\sin(t)\right)^d + (q-1)\left(\sqrt{1+\frac{w}{q}}\cos(t) - \sqrt{\frac{w}{q(q-1)}}\sin(t)\right)^d,$$

then $\Phi_{d,q,w} := \max_{t\in[-\pi,\pi]} \Phi_{d,q,w}(t)$.

If $(G_n)_n$ *is a sequence of random d-regular graphs, then the same statement holds true asymptotically almost surely. In fact, if* $L(G,g)$ *denotes the number of cycles of length at most* $g-1$ *in a graph* G*, then the same conclusion holds if for every fixed* g *we have* $\lim_{n\to\infty} \frac{L(G_n,g)}{v(G_n)} = 0$.

In both theorems there is a phase transition. This is apparent in Theorem 1, where the phase transition is at $x = d-1$. In Theorem 2 there is a phase transition at

$$w_c(d,q) = \frac{q-2}{(q-1)^{1-2/d}-1} - 1.$$

For $w \leq w_c(d,q)$ we have $\Phi_{d,q,w} = \Phi_{d,q,w}(0) = q\left(1+\frac{w}{q}\right)^{d/2}$, and for $w > w_c(d,q)$ we have $\Phi_{d,q,w} > q\left(1+\frac{w}{q}\right)^{d/2}$. Moreover, if $q > 2$, then $\frac{\partial}{\partial w}\Phi_{d,q,w}$ is discontinuous at $w_c(q)$, that is, there is a first order phase transition at $w_c(q)$.

In this lecture note we will give the proof of Theorem 1, and we also give a lot of ingredients and the sketch of the proof or Theorem 2. Furthermore, we present the proof of a slightly easier theorem about the number of Eulerian orientations.

8. Matching polynomial

8.1 Introduction

Definition 8.1.1. Let G be a graph on $v(G) = n$ vertices and let $m_k(G)$ denote the number of matchings of size k. Then the matching polynomial $\mu(G, x)$ is defined as follows:

$$\mu(G,x) = \sum_{k=0}^{\lfloor v/2 \rfloor} (-1)^k m_k(G) x^{n-2k}.$$

Note that $m_0(G) = 1$. Another way to define $\mu(G, x)$ is the following:

$$\mu(G,x) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} x^{n-2|M|}.$$

Sometimes it will be more convenient to work with the so-called matching generating function

$$M(G,\lambda) = \sum_{k=0}^{\lfloor v/2 \rfloor} m_k(G) \lambda^k.$$

This is also the partition function of the monomer-dimer model at fugacity λ.

Clearly, the matching polynomial and the matching generating function encode the same information.

Proposition 8.1.2 ([30, 28])**.** *(a) Let $u \in V(G)$ then*

$$\mu(G,x) = x\mu(G-u,x) - \sum_{v \in N(u)} \mu(G - \{u,v\}, x).$$

(b) For $e = (u, v) \in E(G)$ we have

$$\mu(G,x) = \mu(G-e,x) - \mu(G - \{u,v\}, x).$$

(c) For $G = G_1 \cup G_2 \cup \cdots \cup G_k$ we have

$$\mu(G,x) = \prod_{i=1}^{k} \mu(G_i, x).$$

(d) We have

$$\mu'(G,x) = \sum_{u \in V(G)} \mu(G-u,x).$$

Proof. (a) By comparing the coefficient of x^{n-2k} we need to prove that

$$m_k(G) = m_k(G-u) + \sum_{v \in N(u)} m_{k-1}(G - \{u,v\}).$$

This is indeed true since we can count the number of k-matchings of G as follows: there $m_k(G-u)$ k-matchings which do not contain u, and if a k-matching contains u then there is a unique $v \in N(u)$ such that the edge (u,v) is in the matching, and the remaining $k-1$ edges are chosen from $G - \{u,v\}$.

(b) By comparing the coefficient of x^{n-2k} we need to prove that

$$m_k(G) = m_k(G-e) + m_{k-1}(G - \{u,v\}).$$

This is indeed true since the number of k-matchings not containing e is $m_k(G-e)$, and the number of k-matchings containing $e = (u,v)$ is $m_{k-1}(G - \{u,v\})$.

(c) It is enough to prove the claim when $G = G_1 \cup G_2$, for more components the claim follows by induction. By comparing the coefficient of x^{n-2k} we need to prove that

$$m_k(G) = \sum_{r=0}^{k} m_r(G_1) m_{k-r}(G_2).$$

This is indeed true since a k-matching fof G uniquely determine an r-matching of G_1 and a $(k-r)$-matching of G_2 for some $0 \leq r \leq k$.

(d) This follows from the fact that

$$(m_k(G)x^{n-2k})' = (n-2k)m_k(G)x^{n-1-2k} = \sum_{u \in V(G)} m_k(G-u)x^{n-1-2k}$$

since we can compute the cardinality of the set

$$\{(M,u) \mid u \notin V(M), |M| = k\}$$

in two different ways. □

Theorem 8.1.3 (Heilmann and Lieb [30]). *All zeros of the matching polynomial $\mu(G,x)$ are real.*

Proof. We will prove the following two statements by induction on the number of vertices.

(i) All zeros of $\mu(G,x)$ are real.

(ii) For an x with $\mathrm{Im}(x)>0$ we have

$$\mathrm{Im}\frac{\mu(G,x)}{\mu(G-u,x)}>0$$

for all $u\in V(G)$.

Note that in (ii) we already use the claim (i) inductively, namely that $\mu(G-u,x)$ doesn't vanish for an x with $\mathrm{Im}x>0$. On the other hand, claim (ii) for G implies claim (i). So we need to check claim (i).

By the recursion formula we have

$$\frac{\mu(G,x)}{\mu(G-u,x)}=\frac{x\mu(G-u,x)-\sum_{v\in N(u)}\mu(G-\{u,v\},x)}{\mu(G-u,x)}=x-\sum_{v\in N(u)}\frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)}.$$

By induction we have

$$\mathrm{Im}\frac{\mu(G-u,x)}{\mu(G-\{u,v\},x)}>0$$

for $\mathrm{Im}(x)>0$. Hence

$$-\mathrm{Im}\frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)}>0$$

which gives that

$$\mathrm{Im}\frac{\mu(G,x)}{\mu(G-u,x)}>0.$$

□

Remark 8.1.4. One can also deduce from this proof that the zeros of $\mu(G,x)$ and $\mu(G-u,x)$ interlace each other just like the zeros of a real-rooted polynomial and its derivative.

Definition 8.1.5. Let G be graph with a given vertex u. The *path-tree* $T(G,u)$ is defined as follows. The vertices of $T(G,u)$ are the paths[1] in G which start at the vertex u and two paths joined by an edge if one of them is a one-step extension of the other.

[1]In statistical physics, paths are called self-avoiding walks.

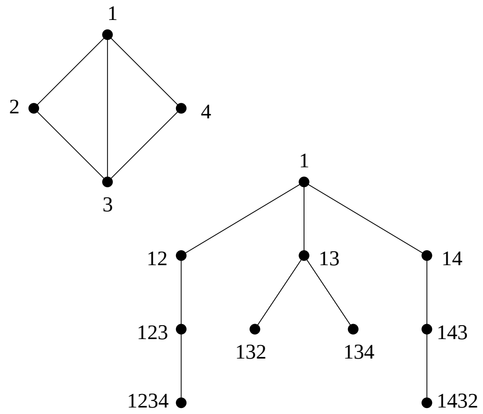


Figure 8.1: A path-tree from the vertex 1.

Proposition 8.1.6. *Let G be a graph with a root vertex u. Let $T(G,u)$ be the corresponding path-tree in which the root is again denoted by u for sake of convenience. Then*

$$\frac{\mu(G-u,x)}{\mu(G,x)} = \frac{\mu(T(G,u)-u,x)}{\mu(T(G,u),x)},$$

and $\mu(G,x)$ divides $\mu(T(G,u),x)$. In fact,

$$\mu(T(G,u),x) = \mu(G,x)\prod_H \mu(H,x)^{\alpha_H},$$

where H are induced subgraphs of G, and α_H is some non-negative integer.

Proof. The proof of this proposition is again by induction using part (a) of Proposition 8.1.2. Indeed,

$$\frac{\mu(G,x)}{\mu(G-u,x)} = \frac{x\mu(G-u,x) - \sum_{v\in N(u)}\mu(G-\{u,v\},x)}{\mu(G-u,x)} =$$

$$= x - \sum_{v\in N(u)}\frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} = x - \sum_{v\in N(u)}\frac{\mu(T(G-u,v)-v,x)}{\mu(T(G-u,v),x)}$$

$$= x\frac{\prod_{v\in N(u)}\mu(T(G-u,v),x) - \sum_{v\in N(u)}\mu(T(G-u,v)-v,x)\prod_{v'\in N(u)\setminus\{v\}}\mu(T(G-u,v'),x)}{\prod_{v\in N(u)}\mu(T(G-u,v),x)}$$

$$= \frac{x\mu(T(G,u)-u,x) - \sum_{v\in N(u)}\mu(T(G,u)-\{u,v\},x)}{\mu(T(G,u)-u,x)} = \frac{\mu(T(G,u),x)}{\mu(T(G,u)-u,x)}.$$

In the first step we used the recursion formula, and in the third step we used the induction step to the graph $G-u$ and root vertex v. Here it is an important observation that $T(G-u,v)$ is exactly the branch of the tree $T(G,u)$ that we get if we delete the vertex u from $T(G,u)$ and consider the subtree rooted at the path uv.

The second part of the claim follows by induction using

$$\mu(T(G,u),x) = \mu(G,x)\frac{\mu(T(G,u)-u,x)}{\mu(G-u,x)}$$

together with the observation that a connected component of $T(G,u)-u$, say rooted at the vertex uv, is path-tree of $G-u$ with root vertex v. □

Proposition 8.1.7 ([30, 28]). *For a forest T, the matching polynomial $\mu(T,x)$ coincides with the characteristic polynomial $\varphi(T,x) = \det(xI - A_T)$.*

Proof. Indeed, when we expand the $\det(xI - A)$ we only get non-zero terms when the cycle decomposition of the permutation consists of cycles of length at most 2; but these terms correspond to the terms of the matching polynomial.

□

Remark 8.1.8. Clearly, Propositions 8.1.6 and 8.1.7 together give a new proof of the Heilmann-Lieb theorem since $\mu(G,x)$ divides $\mu(T(G,u),x) = \varphi(T(G,u),x)$ whose zeros are real since they are the eigenvalues of a symmetric matrix.

Proposition 8.1.9 ([30, 28]). *If the largest degree Δ is at least 2, then all zeros of the matching polynomial lie in the interval $[-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1}]$.*

First proof. First we show that if u is a vertex of degree at most $\Delta-1$, then for any $x \geq 2\sqrt{\Delta-1}$ we have

$$\frac{\mu(G,x)}{\mu(G-u,x)} \geq \sqrt{\Delta-1}.$$

We prove this statement by induction on the number of vertices. This is true if $G = K_1$, so we can assume that $v(G) \geq 2$. Then

$$\begin{aligned}\frac{\mu(G,x)}{\mu(G-u,x)} &= \frac{x\mu(G-u,x) - \sum_{v\in N_G(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)} \\ &= x - \sum_{v\in N_G(u)} \frac{\mu(G-\{u,v\},x)}{\mu(G-u,x)} \geq x - (\Delta-1)\frac{1}{\sqrt{\Delta-1}} \geq \sqrt{\Delta-1}.\end{aligned}$$

We used the fact that $v \in N_G(u)$ has degree at most $\Delta-1$ in the graph $G-u$.

Then for any vertex u we have

$$\frac{\mu(G,x)}{\mu(G-u,x)} = \frac{x\mu(G-u,x) - \sum_{v\in N_G(u)} \mu(G-\{u,v\},x)}{\mu(G-u,x)}$$

$$= x - \sum_{v \in N_G(u)} \frac{\mu(G - \{u, v\}, x)}{\mu(G - u, x)} \geq x - \Delta \frac{1}{\sqrt{\Delta - 1}} > 0$$

since $v \in N_G(u)$ has degree at most $\Delta - 1$ in the graph $G - u$. This shows $\mu(G, x) \neq 0$ if $x \geq 2\sqrt{\Delta - 1}$. Since the zeros of the matching polynomial are symmetric to 0 we get that all zeros lie in the interval $(-2\sqrt{\Delta - 1}, 2\sqrt{\Delta - 1})$. □

Second proof. By Propositions 8.1.6 and 8.1.7 we know that $\mu(G, x)$ divides $\mu(T(G, u), x) = \varphi(T(G, u), x)$. The largest degree of $T(G, u)$ is also at most Δ so we have $\rho_1(T(G, u)) \leq 2\sqrt{\Delta - 1}$. Since the zeros of the matching polynomial are symmetric to 0 we get that all zeros lie in the interval $[-2\sqrt{\Delta - 1}, 2\sqrt{\Delta - 1}]$. □

Proposition 8.1.10. *Let*

$$\frac{\mu(G - u, x)}{\mu(G, x)} = \sum_k a_k(G, u) x^{-(k+1)}.$$

Then $a_k(G, u)$ counts the number of closed walks of length k in the tree $T(G, u)$ from u to u.

Proof. This proposition follows from Proposition 8.1.6 and 8.1.7 and the fact that

$$\frac{\varphi(H - u, x)}{\varphi(H, x)} = \sum_k W_k(H, u) x^{-(k+1)},$$

where $W_k(H, u)$ counts the number of closed walks of length k from u to u in a graph H. Indeed,

$$\frac{\mu(G - u, x)}{\mu(G, x)} = \frac{\mu(T(G, u) - u, x)}{\mu(T(G, u), x)} = \frac{\varphi(T(G, u) - u, x)}{\varphi(T(G, u), x)} = \sum_k W_k(T(G, u), u) x^{-k}.$$

Here $W_k(T(G, u), u) = a_k(G, u)$ by definition. □

Remark 8.1.11. A walk in the tree $T(G, u)$ from u can be imagined as follows. Suppose that in the graph G a worm is sitting at the vertex u at the beginning. Then at each step the worm can either grow or pull back its head. When it grows it can move its head to a neighboring unoccupied vertex while keeping its tail at vertex u. At each step the worm occupies a path in the graph G. A closed walk in the tree $T(G, u)$ from u to u corresponds to the case when at the final step the worm occupies only vertex u. C. Godsil calls these walks tree-like walks in the graph G.

Proposition 8.1.12. *(a) Let*

$$\frac{\mu'(G,x)}{\mu(G,x)} = \sum_k a_k(G) x^{-(k+1)}.$$

Then $a_k(G)$ counts the number of closed tree-like walks of length k.

(b) If $\mu(G,x) = \prod_{i=1}^{v(G)} (x - \alpha_i)$ then for all $k \geq 1$ we have

$$a_k(G) = \sum_{i=1}^{v(G)} \alpha_i^k.$$

9. Subgraph counting polynomial

9.1 One more graph polynomial

In this chapter we always assume that G is a d-regular graph. (One can extend the definition of the subgraph counting polynomial to an arbitrary graph, but in this chapter it will be convenient just to look at this special case.)

Definition 9.1.1. The subgraph counting polynomial is defined as

$$F_G(x_0,\dots,x_d)=\sum_{A\subseteq E}\left(\prod_{v\in V}x_{d_A(v)}\right),$$

and a bit more generally,

$$F_G(x_0,\dots,x_d|z)=\sum_{A\subseteq E}\left(\prod_{v\in V}x_{d_A(v)}\right)z^{2|A|}=F_G(x_0,x_1z,x_2z,...,x_dz^d).$$

As an example we give the subgraph counting polynomial $F_{K_5}(x_0,x_1,x_2,x_3,x_4)$ of the complete graph K_5 on 5 vertices. The first term corresponds to the empty subgraph, the last term corresponds to the graph itself.

$$\begin{aligned}&x_0^5+10x_0^3x_1^2+15x_0x_1^4+30x_0^2x_1^2x_2+30x_1^4x_2+60x_0x_1^2x_2^2+10x_0^2x_2^3+70x_1^2x_2^3+15x_0x_2^4\\&+12x_2^5+20x_0x_1^3x_3+60x_1^3x_2x_3+60x_0x_1x_2^2x_3+120x_1x_2^3x_3+60x_1^2x_2x_3^2+30x_0x_2^2x_3^2+70x_2^3x_3^2\\&+60x_1x_2x_3^3+5x_0x_3^4+30x_2x_3^4+5x_1^4x_4+30x_1^2x_2^2x_4+15x_2^4x_4+60x_1x_2^2x_3x_4+60x_2^2x_3^2x_4\\&+20x_1x_3^3x_4+15x_3^4x_4+10x_2^3x_4^2+30x_2x_3^2x_4^2+10x_3^2x_4^3+x_4^5.\end{aligned}$$

If G is not necessarily d-regular, then the above definitions have to be changed as follows. For each vertex v we introduce a set of variables $x_0^{(v)},x_1^{(v)},\dots,x_{d(v)}^{(v)}$. Then the subgraph counting function is defined as

$$F_G\left(\left(x_0^{(v)},x_1^{(v)},\dots,x_{d(v)}^{(v)}\right)_{v\in V}\right)=\sum_{A\subseteq E}\left(\prod_{v\in V}x_{d_A(v)}^{(v)}\right),$$

and

$$F_G\left(\left(x_0^{(v)}, x_1^{(v)}, \dots, x_{d(v)}^{(v)}\right)_{v\in V} |z\right) = \sum_{A\subseteq E}\left(\prod_{v\in V} x_{d_A(v)}^{(v)}\right) z^{2|A|}.$$

9.1.1 Rank 2 matrices

There are several things that can be encoded by the subgraph counting polynomial. For instance, $F_G(0,1,0,\dots,0)$ counts the number of perfect matchings of the graph G by definition. Later we will show that the number of Eulerian orientations can be encoded by the subgraph counting polynomial too. (See the next chapter.)

In this section we show that even if the matrix N has rank 2 (for instance, because it is itself a 2×2 matrix), then $Z_G(N,\underline{\mu})$ can be encoded by the subgraph counting polynomial.

Suppose that we can write an $r\times r$ matrix N into the form $N = \underline{a}\underline{a}^T + \underline{b}\underline{b}^T$ and let $\underline{\mu}\in\mathbb{R}^r$. Then

$$\begin{aligned}
Z_G(N,\underline{\mu}) &= \sum_{\varphi:V\to[r]}\prod_{v\in V}\mu_{\varphi(v)}\prod_{(u,v)\in E} N_{\varphi(u)\varphi(v)}\\
&= \sum_{\varphi:V\to[r]}\prod_{v\in V}\mu_{\varphi(v)}\prod_{(u,v)\in E}(\underline{a}\underline{a}^T+\underline{b}\underline{b}^T)_{\varphi(u)\varphi(v)}\\
&= \sum_{A\subseteq E}\sum_{\varphi:V\to[r]}\prod_{v\in V}\mu_{\varphi(v)}\prod_{(u,v)\in E\setminus A}(\underline{a}\underline{a}^T)_{\varphi(u)\varphi(v)}\prod_{(u,v)\in A}(\underline{b}\underline{b}^T)_{\varphi(u)\varphi(v)}\\
&= \sum_{A\subseteq E}\sum_{\varphi:V\to[r]}\prod_{v\in V}\mu_{\varphi(v)}\prod_{(u,v)\in E\setminus A}(\underline{a}_{\varphi(u)}\underline{a}_{\varphi(v)})\prod_{(u,v)\in A}(\underline{b}_{\varphi(u)}\underline{b}_{\varphi(v)})\\
&= \sum_{A\subseteq E}\prod_{v\in V}\left(\sum_{k=1}^{r}\mu_k a_k^{d-d_S(v)} b_k^{d_S(v)}\right)\\
&= F_G(r_0,\dots,r_d),
\end{aligned}$$

where $r_j = \sum_{k=1}^r \mu_k a_k^{d-j} b_k^j$. On the other hand, $\underline{a}$ and $\underline{b}$ are not the only vectors satisfying $N = \underline{a}\underline{a}^T + \underline{b}\underline{b}^T$. Indeed, let us define the vectors $\underline{a}(t)$ and $\underline{b}(t)$ as follows:

$$\underline{a}(t)_j = a_j\cos(t) + b_j\sin(t),$$

and

$$\underline{b}(t)_j = -a_j\sin(t) + b_j\cos(t).$$

Then $N = \underline{a}(t)\underline{a}(t)^T + \underline{b}(t)\underline{b}(t)^T$. So each pair $\underline{a}(t), \underline{b}(t)$ gives rise to a vector $\underline{v}(t) = (r_0(t),\dots,r_d(t))$ such that

$$F_G(\underline{v}(t)) = Z_G(N,\underline{\mu}).$$

10. Gauge transformation

10.1 Introduction

In this chapter we discuss a method that enables us to prove complicated combinatorial formulas by algebraic manipulations. First we introduce the concept of the normal factor graph that is a quite general way to encode enumeration problems. Then we study the so-called gauge transformation, a method that provides identities among combinatorial formulas.

10.2 Normal factor graphs and gauge transformations

Definition 10.2.1. A normal factor graph $\mathcal{H} = (V, E, \mathcal{X}, (f_v)_{v \in V})$ is a graph (V, E) equipped with an alphabet $\mathcal{X}$ and a function $f_v : \mathcal{X}^{d_v} \to \mathbb{R}$ at each vertex. At each edge e there is a variable x_e taking values from the alphabet $\mathcal{X}$. The partition function

$$Z(\mathcal{H}) = \sum_{\sigma \in \mathcal{X}^E} \prod_{v \in V} f_v(\sigma_{\partial v}),$$

where $\sigma_{\partial v}$ is the restriction of σ to the the edges incident to the vertex v.

For instance, if $\mathcal{X} = \{0, 1\}$ and

$$f_v(\sigma_1, \dots, \sigma_{d_v}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{d_v} \sigma_i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where d_v is the degree of the vertex v, then $Z(\mathcal{H})$ is exactly the number of perfect matchings of the underlying graph.

Let $\mathcal{H} = (V, E, \mathcal{X}, (f_v)_{v \in V})$ be a normal factor graph with alphabet $\mathcal{X}$. We will show that it is possible to introduce a new normal factor graph $\widehat{\mathcal{H}} = (V, E, \mathcal{Y}, (\widehat{f}_v)_{v \in V})$

on the same graph with new functions $\widehat{f}_v$ and alphabet $\mathcal{Y}$ such that $Z(\widehat{\mathcal{H}}) = Z(\mathcal{H})$. As we will see, sometimes it will be more convenient to study the new normal factor graph $\widehat{\mathcal{H}}$.

Let $\mathcal{Y}$ be a new alphabet, and for each edge $(u,v) \in E$ let us introduce two new matrices, G_{uv} and G_{vu} of size $\mathcal{Y} \times \mathcal{X}$. The new variables will be denoted by $\tau \in \mathcal{Y}^E$, the old ones by $\sigma \in \mathcal{X}^E$. For a vertex v with degree $d_v = k$ let

$$\widehat{f}_v(\tau_{vu_1}, \dots, \tau_{vu_k}) = \sum_{\sigma_{vu_1},\dots,\sigma_{vu_k}} \left(\prod_{u_i \in N(v)} G_{vu_i}(\tau_{vu_i}, \sigma_{vu_i}) \right) f_v(\sigma_{vu_1}, \dots, \sigma_{vu_k}).$$

This way we defined the functions $\widehat{f}_v$ of $\widehat{\mathcal{H}}$.

This transformation is called a gauge transformation. In computer science, this method was introduced by Valiant under the name holographic reduction [49, 52, 51, 50]. In statistical physics, it was developed by Chertkov and Chernyak under the name gauge transformation [18, 19]. Wainwright, Jaakola, Willsky had a related idea under the name reparametrization [55], but it is not easy to see the connection. In the different cases the scope was slightly different, Valiant used it as a reduction method for computational complexity of counting problems. This line of research was extended in a series of papers of Jin-Yi Cai and his coauthors, see Jin-Yi Cai's book [11] and the papers [12, 14, 13, 10, 15, 16] and references therein. Chertkov and Chernyak [18, 19] studied the so-called Bethe–approximation through gauge transformations. We simply use it as a method of proving the identity of Theorem 13.2.1.

The following theorem is due to Chertkov and Chernyak [18, 19] and independently Valiant [49].

Theorem 10.2.2. *If for each edge $(u,v) \in E$ we have $G_{uv}^T G_{vu} = \mathrm{Id}_{\mathcal{X}}$, then $Z(\widehat{\mathcal{H}}) = Z(\mathcal{H})$.*

Proof. Let us start to compute $Z(\widehat{\mathcal{H}}) = \sum_{\tau \in \mathcal{Y}^E} \prod_{v \in V} \widehat{f}_v(\tau_{\partial v})$:

$$Z(\widehat{\mathcal{H}}) = \sum_{\tau \in \mathcal{Y}^E} \prod_{v \in V} \left[\sum_{\sigma_{vu_1},\dots,\sigma_{vu_k}} \left(\prod_{u_i \in N(v)} G_{vu_i}(\tau_{vu_i}, \sigma_{vu_i}) \right) f_v(\sigma_{vu_1}, \dots, \sigma_{vu_k}) \right].$$

If we expand it will have terms $\prod_{v \in V} f_v(\sigma_{vu_1}, \dots, \sigma_{vu_k})$ with some coefficients. A priori it can occur that these terms are incompatible in the sense that $\sigma_{uv} \neq \sigma_{vu}$. As we will see, the role of the conditions on G_{uv} is exactly to ensure that if there is an edge $(u,v) \in E$ with $\sigma_{uv} \neq \sigma_{vu}$, then the coefficient is 0, and if all edges are compatible, then the coefficient is 1. Indeed, the coefficient is

$$\sum_{\tau \in \mathcal{Y}^E} \prod_{v \in V} \prod_{u_i \in N(v)} G_{vu_i}(\tau_{vu_i}, \sigma_{vu_i}).$$

Note that $\tau_{uv} = \tau_{vu}$ for each edge, and this variable appears only at the vertices u and v, and nowhere else. Hence

$$\sum_{\tau \in \mathcal{Y}^E} \prod_{v \in V} \prod_{u_i \in N(v)} G_{vu_i}(\tau_{vu_i}, \sigma_{vu_i}) = \prod_{(u,v) \in E} \left(\sum_{\tau_{uv}} G_{uv}(\tau_{uv}, \sigma_{uv}) G_{vu}(\tau_{vu}, \sigma_{vu}) \right) =$$

$$= \prod_{(u,v) \in E} \left(\sum_{\tau_{uv}} G_{uv}^T(\sigma_{uv}, \tau_{vu}) G_{vu}(\tau_{vu}, \sigma_{vu}) \right) = \prod_{(u,v) \in E} (G_{uv}^T G_{vu})_{\sigma_{uv}, \sigma_{vu}} = \prod_{(u,v) \in E} (\mathrm{Id})_{\sigma_{uv}, \sigma_{vu}}.$$

Hence this is only non-zero if $\sigma_{uv} = \sigma_{vu}$ for each edge $(u,v) \in E(G)$, and then this coefficient is 1. □

10.3 Perfect matchings

As a quick application let see an example.

Theorem 10.3.1. *Let $G = (V,E)$ be a d–regular graph on $2n$ vertices. Then for the number of perfect matchings of G we have*

$$\mathrm{pm}(G) = \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right)^n \sum_{A \subseteq E} \frac{1}{(d-1)^{|A|}} \prod_{v \in V} (1 - d_A(v)),$$

where $d_A(v)$ is the degree of the vertex v in the graph (V,A).

Proof. Clearly, $\mathrm{pm}(G) = \sum_{\sigma \in \mathcal{X}^E} \prod_{v \in V} f_v(\sigma_{\partial v})$, where $\mathcal{X} = \{0,1\}$ and

$$f_v(\sigma_1, \dots \sigma_d) = \begin{cases} 1 & \text{if } \sum_{i=1}^d \sigma_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now let $G_{uv} = G$ for each edge (u,v), where

$$G := \begin{pmatrix} \sqrt{\frac{d-1}{d}} & \frac{1}{\sqrt{d}} \\ \frac{1}{\sqrt{d}} & -\sqrt{\frac{d-1}{d}} \end{pmatrix}.$$

Then $G^T G = \mathrm{Id}$. We have

$$\widehat{f_{\mathbb{G},v}}(0,\dots,0) = dG(0,0)^{d-1}G(0,1) = \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{1/2},$$

and if there are $k \geq 1$ 1's in $\widehat{f_v}$, then

$$\begin{aligned}\widehat{f_v}(1,\dots,1,0,\dots 0) &= kG(1,0)^{k-1}G(1,1)G(0,0)^{d-k} + (d-k)G(1,0)^k G(0,0)^{d-k-1}G(0,1)\\ &= G(1,0)^{k-1}G(0,0)^{d-k-1}(kG(1,1)G(0,0) + (d-k)G(1,0)G(0,1))\\ &= \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{1/2}(1-k)\frac{1}{(d-1)^{k/2}}.\end{aligned}$$

Then the claim follows from $Z(\widehat{\mathcal{H}}) = Z(\mathcal{H})$. □

Remark 10.3.2. Clearly, we can restate the claim as follows:

$$F_G(0,1,0,\dots,0) = \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n F_G\left(1,0,\frac{-1}{d-1},\frac{-2}{(d-1)^{3/2}},\dots,\frac{1-d}{(d-1)^{d/2}}\right).$$

In general, it is true that there are matrices R_t for each $t \in [0,2\pi]$ such that $F_G(\underline{x}) = F_G(R_t\underline{x})$. Such matrices comes from the gauges $G = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$.

10.4 Eulerian orientations

Theorem 10.4.1. *Let $\underline{s} = (s_0, s_1, \dots, s_d)$ be defined as follows.*

$$s_k = \begin{cases} \frac{\binom{d}{d/2}\binom{d/2}{k/2}}{2^{d/2}\binom{d}{k}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Then $F_G(s_0,\dots,s_d)$ counts the number of Eulerian orientations of a d–regular graph G.

Proof. First we encode the number of Eulerian orientations as a partition function of a normal factor graph. Let $\mathrm{Sub}(G)$ be the subdivision of the graph G, that is, we put a vertex to every edge. The vertex set of $\mathrm{Sub}(G)$ naturally correspond to $V \cup E$, where $G = (V,E)$. An orientation of G correspond to an edge configuration of $\mathrm{Sub}(G)$, where each edge $e \in V(\mathrm{Sub}(G))$ is incident to exactly one edge: a directed edge (v,u) corresponds a configuration, where $(v, e_{v,u})$ belongs to the configuration, but $(u, e_{v,u})$ does not. So we can describe an Eulerian orientation with the local functions

$$f_v(\sigma_{v,e_{v,u_1}}, \dots, \sigma_{v,e_{v,u_d}}) = \begin{cases} 1 & \text{if } \sum_{u_i \in N_G(v)} \sigma_{ve_{v,u}} = d/2, \\ 0 & \text{if } \sum_{u_i \in N_G(v)} \sigma_{ve_{v,u}} \neq d/2. \end{cases}$$

and

$$f_{e_{u,v}}(\sigma_{u,e_{u,v}}, \sigma_{v,e_{u,v}}) = \begin{cases} 1 & \text{if } \sigma_{u,e_{u,v}}, \sigma_{v,e_{u,v}} = 1, \\ 0 & \text{if } \sigma_{u,e_{u,v}}, \sigma_{v,e_{u,v}} \neq 1. \end{cases}$$

Next we use the gauge theory. For each edge $e = (u, v) \in E(G)$ we introduce two matrices in $\mathrm{Sub}(G)$: $G_{eu} = G_{ev} = G_1$ and $G_{ue} = G_{ve} = G_2$, where

$$G_1 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{and} \quad G_2 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

In what follows the rows and columns of G_1, G_2, F_e are indexed by 0 and 1, and for a matrix A and $\sigma, \tau \in \{0, 1\}$ we use the notation $A(\sigma, \tau)$ for the corresponding element. In particular, we have

$$F_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Observe that $G_2^T G_1 = \mathrm{Id}$. First let us compute $\widehat{f}_e(\tau_1, \tau_2)$:

$$\begin{aligned} \widehat{f}_e(\tau_1, \tau_2) &= \sum_{\sigma_1, \sigma_2} G_1(\tau_1, \sigma_1) G_1(\tau_2, \sigma_2) f_e(\sigma_1, \sigma_2) \\ &= \sum_{\sigma_1, \sigma_2} G_1(\tau_1, \sigma_1) f_e(\sigma_1, \sigma_2) G_1^T(\sigma_2, \tau_2) \\ &= (G_1 F_e G_1^T)(\tau_1, \tau_2). \end{aligned}$$

Hence by simple matrix multiplication we have

$$\widehat{F_e} = G_1 F_e G_1^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means that in $Z(\widehat{\mathcal{H}})$ only those terms will survive that correspond to a subgraph of G.

Next let us compute $\widehat{f}_v(\tau_1, \dots, \tau_d)$. By definition

$$\widehat{f}_v(\tau_1, \dots, \tau_d) = \sum_{\sigma_1, \dots, \sigma_d} \prod_{i=1}^{d} G_2(\tau_i, \sigma_i) f_v(\sigma_1, \dots, \sigma_d).$$

Recall that only those terms remain, where $\sum_i \sigma_i = \frac{d}{2}$. Suppose that $\sum_i \tau_i = k$. If there are j places where both $\sigma_i = \tau_i = 1$, then its contribution to the sum is $i^j(-i)^{k-j}$, so

$$\widehat{f_v}(\tau_1, \dots, \tau_d) = \sum_{\sigma_1,\dots,\sigma_d} \prod_{i=1}^{d} G_2(\tau_i, \sigma_i) f_v(\sigma_1, \dots, \sigma_d) = \frac{1}{2^{d/2}} \sum_{j=0}^{k} \binom{k}{j} \binom{d-k}{d/2-j} (-1)^{k-j} i^k.$$

Observe that

$$\sum_{j=0}^{d/2} (-1)^{k-j} \binom{k}{j} \binom{d-k}{d/2-j} = \sum_{j=0}^{d/2} (-1)^{k-j} \frac{k!(d-k)!}{j!(k-j)!(d/2-j)!(d/2-k+j)!}$$
$$= \frac{\binom{d}{d/2}}{\binom{d}{k}} \sum_{j=0}^{d/2} (-1)^{k-j} \binom{d/2}{j} \binom{d/2}{k-j}.$$

Note that $\sum_{j=0}^{d/2} (-1)^{k-j} \binom{d/2}{j} \binom{d/2}{k-j}$ is the coefficient of x^k in

$$(1-x)^{d/2}(1+x)^{d/2} = (1-x^2)^{d/2}$$

which is clearly 0 if k is odd, and $(-1)^{k/2}\binom{d/2}{k/2}$ if k is even. Hence

$$\widehat{f_v}(\tau_1, \dots, \tau_d) = s_{||\tau||_1}.$$

This means that

$$Z(\widehat{\mathcal{H}}) = F_G(s_0, \dots, s_d).$$

□

Remark 10.4.2. Let G be a regular graph, and let

$$H_G(y_{-d}, y_{-d+2}, \dots, y_{d-2}, y_d) = \sum_{\mathcal{O}} \prod_{v \in V} y_{d^+_{\mathcal{O}(v)} - d^-_{\mathcal{O}}(v)},$$

where the summation goes for all orientations of the graph G, and $d^+_{\mathcal{O}(v)}$ and $d^-_{\mathcal{O}}(v)$ are the out-degree and in-degree of G in $\mathcal{O}$. In general it is true that there is a $(d+1) \times (d+1)$ matrix M_d such that $H_G(\underline{x}) = F_G(M_d \underline{x})$ for every graph G and $\underline{x} \in \mathbb{C}^{d+1}$. This statement also extends to non-regular graphs.

11. Lee-Yang-type theorems

11.1 Lee-Yang theorem and Ising-model

Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $\overline{D} = \mathbb{C} \setminus D$. In this section we study $\overline{D}$-stable polynomials. A multivariate polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ is $\overline{D}$-stable if $P(z_1, \dots, z_n) \neq 0$ whenever $\min(|z_1|, \dots, |z_n|) > 1$. This is consistent with the general definition of Ω-stability. Our main goal is to prove the following generalization of the Lee-Yang theorem.

Theorem 11.1.1 (Lee-Yang, Asano). *Let G be a graph on vertex set $V(G) = [n]$, and for each edge $(u, v) \in E(G)$ let $a_{u,v}$ be a real number satisfying $|a_{u,v}| \leq 1$. Then the polynomial*

$$J_G(z_1, \dots, z_n) = \sum_{S \subseteq V(G)} \left(\prod_{\substack{(u,v) \in E(G) \\ u \in S, v \in \overline{S}}} a_{u,v} \right) \prod_{u \in S} z_u$$

is $\overline{D}$-stable.

Remark 11.1.2. Sometimes the above theorem phrased as follows. Let $A = (a_{ij})$ be a symmetric matrix such that $|a_{ij}| \leq 1$ for all $i, j \in [n]$. Then the polynomial

$$J(z_1, \dots, z_n) = \sum_{S \subseteq [n]} \left(\prod_{\substack{i \in S \\ j \in \overline{S}}} a_{i,j} \right) \prod_{i \in S} z_i$$

is $\overline{D}$-stable.

From this theorem the above version can be obtained by writing 1 to the non-edges. This version can be obtained by applying the above theorem to the complete graph.

The theorem remains true if $a_{i,j}$ are complex numbers, but in this case $a_{j,i} = \overline{a_{i,j}}$ should hold for all $i, j \in [n]$. The whole proof remains valid, only we need to change $1 + \beta(z_1 + z_2) + z_1 z_2$ to $1 + \beta z_1 + \overline{\beta} z_2 + z_1 z_2$ in Lemma 11.1.7 below.

Before we start to prove Theorem 11.1.1 let us see some important consequences.

Theorem 11.1.3 (Lee-Yang)**.** *Let G be a graph on vertex set, and for each edge $(u,v) \in E(G)$ let $a_{u,v}$ be a real number satisfying $|a_{u,v}| \leq 1$. Let*

$$P_G(z) = \sum_{S \subseteq V(G)} \left(\prod_{\substack{(u,v)\in E(G) \\ u\in S, v\in \overline{S}}} a_{u,v} \right) z^{|S|}$$

Then all zeros of the polynomial $P_G(z)$ have absolute value 1.

Proof. Note that

$$P_G(z) = J_G(z, z, \dots, z).$$

Hence it cannot have a zero z with $|z| > 1$. On the other hand, the coefficient of the term corresponding to S is the same as the one corresponding to $\overline{S}$ whence

$$J_G(z_1, \dots, z_n) = \left(\prod_{i=1}^{n} z_i \right) J_G \left(\frac{1}{z_1}, \dots, \frac{1}{z_n} \right).$$

From this it follows that $P_G(z) \neq 0$ if $0 < |z| < 1$. Finally, $P_G(0) = 1 \neq 0$. Hence all zeros of $P_G(z)$ have absolute value 1. □

Theorem 11.1.4. *Let $\beta > 0$. Consider the partition function of the Ising-model with $\nu(1) = z, \nu(-1) = \frac{1}{z}$:*

$$Z(G, A_{\mathrm{Is}(\beta)}, \nu_{\ln(z)}) = \sum_{\sigma: V(G) \to \{-1,1\}} \exp \left(\sum_{(u,v)\in E(G)} \beta\sigma(u)\sigma(v) + \ln(z) \sum_{u \in V(G)} \sigma(u) \right).$$

Then all zeros of $Z(G, A_{\mathrm{Is}(\beta)}, \nu_{\ln(z)})$ have absolute value 1.

Proof. Let $e(G)$ be the number of edges of G, and set $a_{u,v} = e^{-2\beta}$ if $(u,v) \in E(G)$. Then with $P_G(z)$ defined in Theorem 11.1.3 we have

$$Z(G, A_{\mathrm{Is}(\beta)}, \nu_{\ln(z)}) = z^{-v(G)} e^{e(G)\beta} P_G(z^2).$$

Hence the theorem follows from Theorem 11.1.3.

□

Now let us start to prove Theorem 11.1.1. The main tool of the its proof will be the so-called Asano contraction.

Lemma 11.1.5 (Asano)**.** *Suppose that the affine polynomial*

$$P(z_1, z_2, z_3, \dots z_n) = A(z_3, \dots, z_n) + B(z_3, \dots, z_n)z_1 + C(z_3, \dots, z_n)z_2 + D(z_3, \dots, z_n)z_1z_2$$

is $\overline{D}$*-stable and* D *is not the constant* 0 *polynomial. Then the polynomial*

$$Q(z, z_3, \dots, z_n) = A(z_3, \dots, z_n) + D(z_3, \dots, z_n)z.$$

is $\overline{D}$*-stable too.*

Remark 11.1.6. The condition that D is not the constant 0 polynomial is necessary. Let $|\beta| < 1$ and consider the polynomial

$$P(z_1, z_2, z_3) = (1 + \beta z_3) + (\beta + z_3)z_1.$$

This is $\overline{D}$-stable as we will see. On the other hand, $A(z_3) = 1 + \beta z_3$ is not $\overline{D}$-stable.

Proof. Let us fix some $z_3, \dots, z_n$ such that $\min(|z_3|, \dots, |z_n|) > 1$, and set

$$A_0 = A(z_3, \dots, z_n),\ B_0 = B(z_3, \dots, z_n),\ C_0 = C(z_3, \dots, z_n), D_0 = D(z_3, \dots, z_n).$$

The polynomial $A_0 + (B_0 + C_0)z + D_0z^2$ cannot have a zero of absolute value more than 1, because this would contradict the stability of P. Suppose that $D_0 \neq 0$. If the zeros of the polynomial $A_0 + (B_0 + C_0)z + D_0z^2$ are γ_1, γ_2 then

$$\left|\frac{A_0}{D_0}\right| = |\gamma_1\gamma_2| \leq 1.$$

Then $Q(z, z_3, \dots, z_n)$ cannot be 0 if $|z| > 1$.

Now suppose that $D_0 = 0$. If $Q(z, z_3, \dots, z_n) = 0$ then A_0 must be 0. In this case we show that P cannot be $\overline{D}$-stable. If B_0 and C_0 are both 0 then we can choose any z_1, z_2 having absolute value bigger than 1 to show that P is not $\overline{D}$-stable. Suppose now that at least one of B_0 and C_0 are not 0. By symmetry we can assume that $B_0 \neq 0$. Since the polynomials D is not constant 0 we can choose a $(z'_3, \dots, z'_n)$ in the neighborhood of $(z_3, \dots, z_n)$ such that

$$0 < |D(z'_3, \dots, z'_n)| < |B(z'_3, \dots, z'_n)|.$$

This can be achieved since $D_0 = 0$ and $B_0 \neq 0$ and these functions are continuous. Let

$$A_1 = A(z'_3, \dots, z'_n),\ B_1 = B(z'_3, \dots, z'_n),\ C_1 = C(z'_3, \dots, z'_n), D_0 = D(z'_3, \dots, z'_n).$$

Set

$$z_2 = -\frac{A_1 + B_1 z_1}{C_1 + D_1 z_1}$$

for some z_1 with large absolute value. Here $C_1 + D_1 z_1 \neq 0$ if $|z_1| > |C_1|/|D_1|$. Then

$$|z_2| \geq \frac{|B_1 z_1| - |A_1|}{|C_1| + |D_1||z_1|} > \frac{|B_1|}{|D_1|} - \varepsilon > 1$$

for z_1 with large enough $|z_1|$. But then $P(z_1, z_2, z_3', \dots, z_n') = 0$ contradicting the assumption. We are done. □

Lemma 11.1.7. *Let β be a real number satisfying $|\beta| \leq 1$, and let z_1, z_2 be complex numbers such that $\min(|z_1|, |z_2|) > 1$. Then*

$$1 + \beta(z_1 + z_2) + z_1 z_2 \neq 0.$$

In other words, the polynomial $P(x_1, x_2) = 1 + \beta(x_1 + x_2) + x_1 x_2$ is $\overline{D}$-stable.

Proof. Suppose for contradiction that

$$1 + \beta(z_1 + z_2) + z_1 z_2 = 0.$$

Then

$$z_2 = \frac{1 + \beta z_1}{\beta + z_1}.$$

Note that the denominator cannot be 0 since $|\beta| \leq 1 < |z_1|$. Then

$$|z_2|^2 = \frac{|1 + \beta z_1|^2}{|\beta + z_1|^2} = \frac{(1 + \beta z_1)(1 + \overline{\beta z_1})}{(\beta + z_1)(\overline{\beta} + \overline{z_1})} = \frac{1 + |\beta|^2|z_1|^2 + 2\mathrm{Re}(\beta z_1)}{|\beta|^2 + |z_1|^2 + 2\mathrm{Re}(\beta \overline{z_1})}.$$

Since β is real we have $\mathrm{Re}(\beta z_1) = \mathrm{Re}(\beta \overline{z_1})$. Since $|z_2| > 1$ we get that

$$1 + |\beta|^2|z_1|^2 + 2\mathrm{Re}(\beta z_1) > |\beta|^2 + |z_1|^2 + 2\mathrm{Re}(\beta \overline{z_1}).$$

This is equivalent with

$$(1 - |\beta|^2)(1 - |z_1|^2) > 0$$

which contradicts the assumption that $|\beta| \leq 1$ and $|z_1| > 1$. □

Proof of Theorem 11.1.1. Let u, v be the vertices of a graph H and let H^* be a graph obtained from H by contracting u and v to a new vertex t such a way that for a vertex s let $a_{t,s} = a_{t,u} a_{t,v}$, where $a_{x,y} = 1$ if $(x, y) \notin E(H)$. Then if

$$J_H(\underline{z}) = A + B z_u + C z_v + D z_u z_v$$

then

$$J_{H^*}(\underline{z}) = A + Dz_t.$$

Indeed, A, B, C, D correspond to the cases where u and v are in S or not in S, and we can think to H^* as u and v be forced to be in S or not in S at the same time. Note that D is not the constant 0 polynomial as $S = V(H)$ shows that $\prod_{v \in V(H)} z_v$ is a term in the polynomial. This shows that if J_H is $\overline{D}$-stable then so is J_{H^*}.

We show that G can be obtained from subsequent contractions of a graph H having a $\overline{D}$-stable J_H. Let H be the graph with $2|E(G)| + |V(G)|$ vertices, and $|E(G)|$ edges as follows. For each $(u, v) \in E(G)$ we introduce two vertices $w_{(u,v)}$ and $w_{(v,u)}$ with variables $z_{(u,v)}$ and $z_{(v,u)}$, and put an edge between them with value $a_{u,v}$. The remaining $|V(G)|$ vertices are isolated vertices and we associate a variable z_u corresponding to $u \in V$.

Observe that if H_1 and H_2 are two graphs on disjoint vertex set then

$$J_{H_1 \cup H_2} = J_{H_1} \cdot J_{H_2}.$$

Consequently for the graph H we get that

$$J_H = \prod_{(u,v) \in E(G)} (1 + a_{u,v} z_{(u,v)} + a_{v,u} z_{(v,u)} + z_{(u,v)} z_{(v,u)}) \cdot \prod_{u \in V(G)} (1 + z_u).$$

According to Lemma 11.1.7 this is a $\overline{D}$-stable polynomial. Now let us start to contract the vertices $w_{(u,v)}$ with vertex u for all $(u, v) \in E(G)$. In each step we call the new variable z_u again. Since the polynomial is multivariate linear, it will not cause any problem.

Then at the end we get the graph G, and we get that the corresponding polynomial J_G is $\overline{D}$-stable.

□

11.2 Wagner's theorem

In this section we give a theorem of Wagner (Theorem 3.2 of [54]) without proof about the location of zeros of $F_G(x_0, \dots, x_d | z)$. For any fixed vertex v and $x_0^{(v)}, \dots, x_{d(v)}^{(v)}$ let us define the following *key-polynomial*

$$K_v(x_0^{(v)}, \dots, x_d^{(v)} | z) = \sum_{k=0}^{d(v)} \binom{d(v)}{k} x_k^{(v)} z^k.$$

Theorem 11.2.1 (Wagner [54]). *If for any vertex v the polynomial $K_v(x_0^{(v)}, \dots, x_d^{(v)}|z)$ has no complex zero in the open disk of radius κ around 0, then $F_G\left(\left(x_0^{(v)}, x_1^{(v)}, \dots, x_{d(v)}^{(v)}\right)_{v\in V}|z\right)$ has no complex zero in the open disk of radius κ around 0 for any d-regular graph G.*

If for any vertex v the polynomial $K_v(x_0^{(v)}, \dots, x_d^{(v)}|z)$ has no complex zero in the complement of a closed disk of radius κ around 0, then $F_G\left(\left(x_0^{(v)}, x_1^{(v)}, \dots, x_{d(v)}^{(v)}\right)_{v\in V}|z\right)$ has no complex zero in the complement of a closed disk of radius κ around 0 for any graph G.

In particular, if for any vertex v the polynomial $K_v(x_0^{(v)}, \dots, x_d^{(v)}|z)$ has only zeros on the circle of radius κ around 0, then $F_G\left(\left(x_0^{(v)}, x_1^{(v)}, \dots, x_{d(v)}^{(v)}\right)_{v\in V}|z\right)$ has complex zeros only on the circle of radius κ for any graph G.

12. Combinatorial Approximation I

12.1 Introduction

In this section we introduce various approximations of the partition function of the random cluster model. In the sequel the rank 2 approximation will be especially important for us. In this chapter we use the notation $Z_G(q, w)$ for $Z_{\mathrm{RC}}(G, q, w) = \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|}$.

12.2 Rank 1 approximation.

For motivational purposes let us assume for a moment that q is a positive integer. Then it is known that

$$Z_G(q, w) = Z_G(M),$$

where M is the $q \times q$ matrix with entries $1+w$ in the diagonal and 1's as off-diagonal elements. It is a natural idea to approximate M with the rank 1 matrix M_1 such that the sum of all entries of M and M_1 are equal. In other words, let M_1 be the $q \times q$ matrix with entries $1 + \frac{w}{q}$ everywhere. Note that by the definition of $Z_G(M_1)$ we have

$$Z_G(M_1) = q^{v(G)} \left(1 + \frac{w}{q}\right)^{e(G)}.$$

Let us call the quantity

$$Z_G^{(1)}(q, w) = q^{v(G)} \left(1 + \frac{w}{q}\right)^{e(G)}$$

the rank 1 approximation of $Z_G(q, w)$. This quantity makes sense even if q is positive, but not necessarily integer and we will refer to it as the rank 1 approximation of $Z_G(q, w)$ even in this case.

Lemma 12.2.1. *If $q \geq 1$, then*

$$Z_G(q,w) \geq Z_G^{(1)}(q,w).$$

If $0 < q \leq 1$, then

$$Z_G(q,w) \leq Z_G^{(1)}(q,w).$$

Proof. Using the fact that $k(A) \geq v(G) - |A|$ for an $A \subseteq E(G)$ we get that for $q \geq 1$ we have

$$Z_G(q,w) = \sum_{A \subseteq E(G)} q^{k(A)} w^{|A|} \geq \sum_{A \subseteq E(G)} q^{v(G)-|A|} w^{|A|} = q^{v(G)} \left(1 + \frac{w}{q}\right)^{e(G)}.$$

For $q \leq 1$ we have the opposite inequality in the above computation. □

12.3 Rank 2 approximation

What is better than a rank 1 approximation? Naturally, a rank 2 approximation.

Again for motivational purposes let us assume for a moment that $q \geq 2$ is an integer. This time let us approximate the matrix M with the following rank 2 matrix M_2.

$$M_2 = \begin{pmatrix} 1+w & 1 & \dots & 1 \\ 1 & 1+\frac{w}{q-1} & \dots & 1+\frac{w}{q-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1+\frac{w}{q-1} & \dots & 1+\frac{w}{q-1} \end{pmatrix}.$$

Then

$$Z_G(M_2) = \sum_{S \subseteq V} (1+w)^{e(S)} (q-1)^{v(G)-|S|} \left(1 + \frac{w}{q-1}\right)^{e(G-S)}.$$

Indeed, let $S = \varphi^{-1}(1)$ in the definition of $Z_G(M_2)$. Let us introduce the quantity

$$Z_G^{(2)}(q,w) = \sum_{S \subseteq V} (1+w)^{e(S)} (q-1)^{v(G)-|S|} \left(1 + \frac{w}{q-1}\right)^{e(G-S)}.$$

The definition of $Z_G^{(2)}(q,w)$ makes perfect sense if $q > 1$, but not necessarily integer and we will refer to it as the rank 2 approximation of $Z_G(q,w)$. Recall that

$$M_2' = \begin{pmatrix} 1+w & 1 \\ 1 & 1+\frac{w}{q-1} \end{pmatrix} \quad \text{and} \quad \underline{\nu}_2 = \begin{pmatrix} 1 \\ q-1 \end{pmatrix},$$

and note that

$$Z_G^{(2)}(q,w) = Z_G(M_2', \underline{\nu}_2)$$

even if q is not an integer.

This time it is less clear that it is a natural approximation, but as it will turn out this is an asymptotically precise approximation for essentially large girth graphs if $q \geq 2$ and $w \geq 0$. We can prove it through a series of lemmas.

Lemma 12.3.1. *We have*

$$Z_G(q,w) = \sum_{S \subseteq V} (1+w)^{e(S)} Z_{G-S}(q-1,w).$$

Proof. This identity is trivially true for positive integer q using the interpretation of $Z_G(q,w)$ as the partition function of the Potts-model. Since we have polynomials on both sides we get that it is true for all q and w. □

Lemma 12.3.2. *For $q \geq 2$ we have*

$$Z_G(q,w) \geq Z_G^{(2)}(q,w).$$

For $1 < q \leq 2$ we have

$$Z_G(q,w) \leq Z_G^{(2)}(q,w).$$

Proof. By Lemma 12.3.1 we have

$$Z_G(q,w) = \sum_{S \subseteq V} (1+w)^{e(S)} Z_{G-S}(q-1,w).$$

By the definitions of $Z_G^{(2)}(q,w)$ and $Z_G^{(1)}(q,w)$ we have

$$Z_G^{(2)}(q,w) = \sum_{S \subseteq V} (1+w)^{e(S)} Z_{G-S}^{(1)}(q-1,w).$$

Now the claim follows by Lemma 12.2.1 □

Now we are ready to prove that the rank 2 approximation is asymptotically precise for essentially large girth graphs if $q \geq 2$ and $w \geq 0$.

Theorem 12.3.3. *Let G be a graph on n vertices with $L = L(G,g)$ cycles of length at most $g-1$. Let $q \geq 2$. Then*

$$Z_G^{(2)}(q,w) \leq Z_G(q,w) \leq q^{n/g+L} Z_G^{(2)}(q,w).$$

Proof. The lower bound was already proven in Lemma 12.3.2. So we only need to prove the upper bound.

Given $A \subseteq E(G)$ we can decompose A as follows. Let $V_1, \dots, V_r$ be the vertex sets of the connected components of the graph $H = (V, A)$, and let $A_1, \dots, A_r$ be the corresponding subsets of A. If V_i is an isolated vertex, then $A_i = \emptyset$.

Let us say that V_i is small if the induced graph $G[V_i]$ does not contain a cycle. In particular, A_i does not contain a cycle either. Note that it is possible that A_i does not contain a cycle, but the induced graph $G[V_i]$ contains a cycle, and so V_i is not small. Let $\mathcal{S}_A$ denote the set of small V_i's. We say that V_i is large if it is not small, and we denote by $\mathcal{L}_A$ the set of large V_i's. Note that $|\mathcal{L}_A| \leq n/g + L$ since each large connected component has size at least g or it contains a cycle of length at most $g-1$.

Finally, let us say that a vertex set R is compatible with A if R is the union of some small V_i's. Note that R may be the empty set. We denote this relation by $R \sim A$. Furthermore, let $A[\![R]\!]$ be the edges of A induced by the vertex set R. Note that if $R \sim A$, then $A[\![R]\!]$ is a forest. On the other hand, there is no restriction on $A[\![V \setminus R]\!]$. Figure 12.1 depicts an example for the introduced concepts.

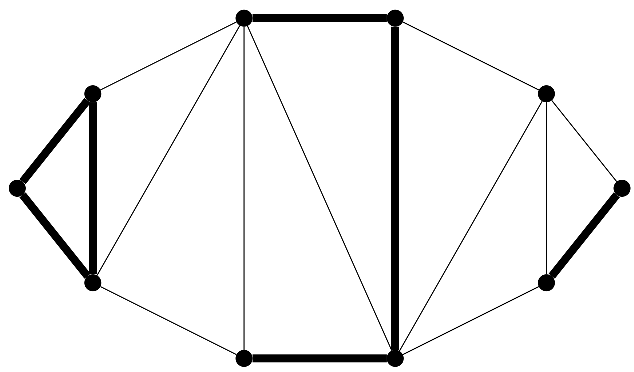


Figure 12.1: A subgraph A is depicted with thick edges. There are 4 components. The edge sets with connected components of size 3 and 4 belong to A_ℓ. The edge sets with connected components of size 1 and 2 belong to A_s. A compatible set R is either the vertex set of the latter components or the empty set or the union of these two connected components.

Let $k(R, A[\![R]\!])$ denote the number of connected components of the graph $(R, A[\![R]\!])$. By the binomial identity we have

$$q^{|\mathcal{S}_A|} = ((q-1)+1)^{|\mathcal{S}_A|} = \sum_{R \sim A} (q-1)^{k(R, A[\![R]\!])}.$$

Then

$$\begin{aligned}
Z_G(q,w) &= \sum_{A\subseteq E(G)} q^{k(A)} w^{|A|} \\
&= \sum_{A\subseteq E(G)} q^{|\mathcal{S}_A|+|\mathcal{L}_A|} w^{|A|} \\
&\leq q^{n/g+L} \sum_{A\subseteq E(G)} q^{|\mathcal{S}_A|} w^{|A|} \\
&= q^{n/g+L} \sum_{A\subseteq E(G)} \sum_{R:R\sim A} (q-1)^{k(R,A[\![R]\!])} w^{|A|} \\
&= q^{n/g+L} \sum_{R\subseteq V(G)} \sum_{A:R\sim A} (q-1)^{k(R,A[\![R]\!])} w^{|A[\![R]\!]|+|A[\![V\setminus R]\!]|} \\
&= q^{n/g+L} \sum_{R\subseteq V(G)} (1+w)^{e(V\setminus R)} \sum_{D} (q-1)^{k(R,D)} w^{|D|},
\end{aligned}$$

where in the last sum, $D = A[\![R]\!]$ is a subset of the edges induced by R such that none of the induced connected components contains a cycle. Then

$$\sum_D (q-1)^{k(R,D)} w^{|D|} = \sum_D (q-1)^{|R|-|D|} w^{|D|} \leq (q-1)^{|R|} \left(1+\frac{w}{q-1}\right)^{e(R)}.$$

Hence

$$Z_G(q,w) \leq q^{n/g+L} \sum_{R\subseteq V(G)} (1+w)^{e(V\setminus R)} Z^{(1)}_{G[R]}(q-1,w),$$

that is

$$Z_G(q,w) \leq q^{n/g+L} Z^{(2)}_G(q,w).$$

□

The following theorem is an immediate consequence of Theorem 12.3.3.

Theorem 12.3.4. *Let $q \geq 2$ and $w \geq 0$. Let $(G_n)_n$ be an essentially large girth sequence of d-regular graphs. If the limit*

$$\lim_{n\to\infty} \frac{1}{v(G_n)} \ln Z^{(2)}_{G_n}(q,w)$$

exists, then the limit

$$\lim_{n\to\infty} \frac{1}{v(G_n)} \ln Z_{G_n}(q,w)$$

exists too, and they have the same value.

13. Combinatorial approximation II

13.1 The polynomial $R_G(z)$

In this section, we study a polynomial that is related to the matching polynomial of a graph G. Recall that a set of edges M is a matching if no two edges of M have common end vertices. A k-matching is simply a matching of size k.

Definition 13.1.1. Let

$$R_G(z) = \sum_{M \in \mathcal{M}(G)} (-z)^{|M|} \prod_{v \notin V(M)} (z + d_v - 1),$$

where d_v is the degree of the vertex v, and $\mathcal{M}(G)$ is the set of matchings of G including the empty one.

In what follows, we first study $R_G(z)$ from the perspective of a model that we call the half-edge model, and is inspired by the monomer-dimer model of statistical physics. This perspective will enable us to rewrite $R_G(z+1)$ as a weighted sum of pseudo-forests.

13.2 Half-edge model

In this part, we introduce the half-edge model. A half-edge configuration is a configuration of edges and half-edges of the graph G such that each vertex of G is incident to at most one edge or half-edge. For such a configuration C, let C_0 be the number of edges of G, where no half edge is chosen, C_1 is the number of edges, where exactly one half is chosen, and C_2 where both halves are chosen (in other words, the edge is chosen). Let

$$M_G(a_0, a_1, a_2) = \sum_C a_0^{C_0} a_1^{C_1} a_2^{C_2}.$$

An equivalent way to think about a half-edge configuration is to consider the partial orientation of the edges, where every vertex has out-degree at most 1. To see this, consider a half-edge configuration A. If a half-edge is chosen at a vertex v, then orient the edge away from v (we allow edges to have double orientation). This is clearly a natural bijection between partial orientations with out-degree at most 1 and half-edge configurations.

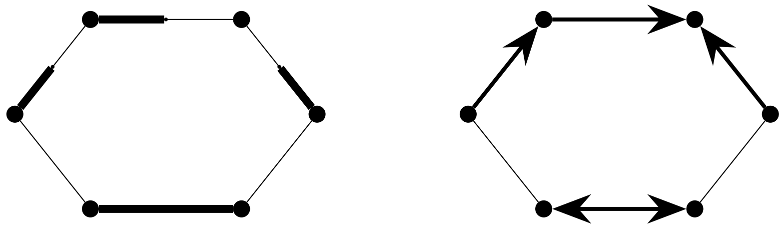


Figure 13.1: A 6-cycle with a half-edge configuration contributing a term $a_0^2 a_1^3 a_2$ and its partial orientation representation.

Theorem 13.2.1. *For a graph G on n vertices we have*

$$M_G(a_0, a_1, a_2) = a_0^{|E|-n} \sum_{M \in \mathcal{M}(G)} (a_0 a_2 - a_1^2)^{|M|} \prod_{v \notin V(M)} (a_0 + d_v a_1),$$

where d_v is the degree of the vertex v for all $v \in V$.

Proof. First we turn the half-edge model into a normal factor graph. First let $\mathrm{Sub}(G) = (V', E')$ be the subdivision of the graph G, that is, we subdivide each edge of the graph with a new vertex. A new vertex will be denoted by e too just like the edge of G that we subdivided, this abuse of the notation will not cause any confusion. Similarly, we will use the notation v for both a vertex of the original graph G and the corresponding vertex in $\mathrm{Sub}(G)$. With this abused notation we have $V' = V \cup E$.

Then let $\mathcal{H}$ be the normal factor graph with underlying graph $\mathrm{Sub}(G)$, alphabet $\mathcal{X} = \{0, 1\}$ and the following functions. For a vertex v with degree d_v corresponding to a vertex of the original graph let

$$f_v(\sigma_1, \dots, \sigma_{d_v}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{d_v} \sigma_i \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

For the new vertices

$$f_e(\sigma_1, \sigma_2) = a_{\sigma_1 + \sigma_2}.$$

Later it will be more convenient to work with the matrix

$$F_e := \begin{pmatrix} a_0 & a_1 \\ a_1 & a_2 \end{pmatrix}.$$

Then $Z(\mathcal{H}) = M_G(a_0, a_1, a_2)$.

Next we use the gauge theory. For each edge $e = (u, v) \in E(G)$ we introduce two matrices in $\mathrm{Sub}(G)$: $G_{eu} = G_{ev} = G_1$ and $G_{ue} = G_{ve} = G_2$, where

$$G_1 := \begin{pmatrix} 1 & 0 \\ -\frac{a_1}{a_0} & 1 \end{pmatrix} \quad \text{and} \quad G_2 := \begin{pmatrix} 1 & \frac{a_1}{a_0} \\ 0 & 1 \end{pmatrix}.$$

In what follows the rows and columns of G_1, G_2, F_e are indexed by 0 and 1, and for a matrix A and $\sigma, \tau \in \{0, 1\}$ we use the notation $A(\sigma, \tau)$ for the corresponding element.

Observe that $G_2^T G_1 = \mathrm{Id}$. First let us compute $\widehat{f}_e(\tau_1, \tau_2)$:

$$\begin{aligned}
\widehat{f}_e(\tau_1, \tau_2) &= \sum_{\sigma_1, \sigma_2} G_1(\tau_1, \sigma_1) G_1(\tau_2, \sigma_2) f_e(\sigma_1, \sigma_2) \\
&= \sum_{\sigma_1, \sigma_2} G_1(\tau_1, \sigma_1) f_e(\sigma_1, \sigma_2) G_1^T(\sigma_2, \tau_2) \\
&= (G_1 F_e G_1^T)(\tau_1, \tau_2).
\end{aligned}$$

Here we have

$$G_1 F_e G_1^T = \begin{pmatrix} a_0 & 0 \\ 0 & \frac{a_0 a_2 - a_1^2}{a_0} \end{pmatrix}.$$

Next let us compute $\widehat{f}_v(\tau_1, \dots, \tau_{d_v})$, where d_v is the degree of the vertex v. By definition

$$\widehat{f}_v(\tau_1, \dots, \tau_{d_v}) = \sum_{\sigma_1, \dots, \sigma_{d_v}} \prod_{i=1}^{d_v} G_2(\tau_i, \sigma_i) f_v(\sigma_1, \dots, \sigma_{d_v}).$$

In particular,

$$\widehat{f}_v(0, \dots, 0) = G_2(0, 0)^{d_v} + d_v G_2(0, 0)^{d_v - 1} G_2(0, 1) = 1 + d_v \frac{a_1}{a_0} = \frac{a_0 + d_v a_1}{a_0}$$

and

$$\widehat{f}_v(1, 0, \dots, 0) = G_2(0, 0)^{d_v - 1} G_2(1, 1) = 1,$$

and

$$\widehat{f}_v(\tau_1, \dots, \tau_{d_v}) = 0$$

if $\sum_{i=1}^{d_v} \tau_i \geq 2$. To see the last two evaluations observe that since $G_2(1,0) = 0$, a non-zero term implies that if $\tau_i = 1$, then $\sigma_i = 1$ too. Hence in case of a non-zero term $\sum_{i=1}^{d_v} \sigma_i \geq \sum_{i=1}^{d_v} \tau_i$. Since $f_v(\sigma_1, \dots \sigma_{d_v}) = 0$ if $\sum_{i=1}^{d_v} \sigma_i \geq 2$ we only need to check a few terms.

Now let us compute

$$Z(\widehat{\mathcal{H}}) = \sum_{\tau \in \mathcal{Y}^{E'}} \prod_{v \in V} \widehat{f}_v(\tau_{\partial v}) \prod_{e \in E} \widehat{f}_e(\tau_{\partial e}).$$

Since $\widehat{f}_e(0,1) = \widehat{f}_e(1,0) = 0$, the non-zero terms correspond to the edge set of the original graph G. Furthermore, since $\widehat{f}_v(\tau_1, \dots, \tau_{d_v}) = 0$ if $\sum_{i=1}^{d_v} \tau_i \geq 2$ this edge set has to be a matching. The contribution of a matching M is

$$\left(\frac{a_2 a_0 - a_1^2}{a_0}\right)^{|M|} a_0^{|E|-|M|} \prod_{v \notin V(M)} \frac{a_0 + d_v a_1}{a_0} = a_0^{|E|-n} (a_2 a_0 - a_1^2)^{|M|} \prod_{v \notin V(M)} (a_0 + d_v a_1).$$

Hence

$$Z(\widehat{\mathcal{H}}) = a_0^{|E|-n} \sum_{M \in \mathcal{M}(G)} (a_0 a_2 - a_1^2)^{|M|} \prod_{v \notin V(M)} (a_0 + d_v a_1).$$

Since $Z(\mathcal{H}) = Z(\widehat{\mathcal{H}})$ the claim of the theorem follows. $\square$

An immediate corollary is the following.

Corollary 13.2.2. *We have*

$$M_G(z, 1, -1) = z^{|E|-n} R_G(z+1).$$

Next, we prove an alternative description of $M_G(z, 1, -1)$.

Definition 13.2.3. Given a graph $G = (V, E)$ and an $A \subseteq E(G)$ we say that A is pseudo-forest of the graph G if each of its connected components are forests or unicyclic graphs. Let $\mathcal{PF}(G)$ be the set of pseudo-forests of the graph G. For a pseudo-forest A let $c(A)$ be the number of cycles in A.

Lemma 13.2.4. *We have*

$$M_G(z, 1, -1) = z^{|E|-n} \sum_{k=0}^{n} \left(\sum_{\substack{A \in \mathcal{PF}(G) \\ |A| = k}} 2^{c(A)} \right) z^{n-k}.$$

Proof. Let us fix a half-edge configuration. Let A be the subset of the edges where at least one of the half-edges is chosen. In other words, if we consider the equivalent partial orientation, then we simply take the oriented edges and we forget the orientation to get A.

Consider a connected component of A, let it be T and let $V(T)$ be the vertices covered by T. Since at each vertex in $V(T)$ there is at most one half-edge chosen in T, we get that $|T| \leq |V(T)|$. On the other hand, T is connected and so $|T| \geq |V(T)|-1$. Now let us calculate the total weight of those configurations, where T is a connected component of the configuration.

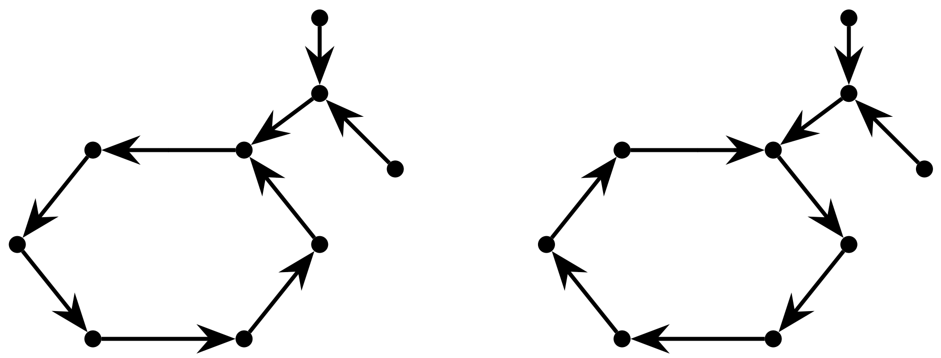


Figure 13.2: Two possible half-edge configurations when A contains a cycle. They only differ in the orientation of the cycle. Every other edge is oriented toward the cycle.

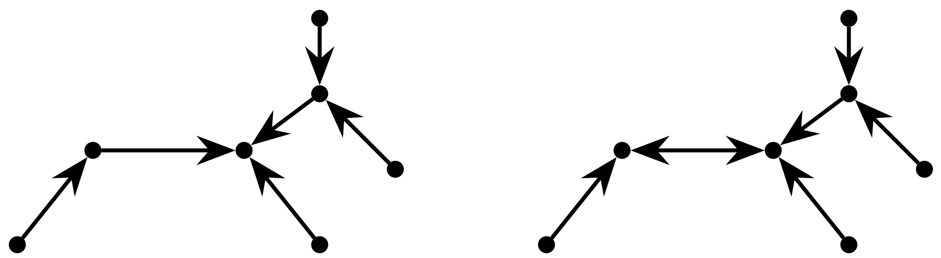


Figure 13.3: Two type of half-edge configurations when A does not contain a cycle. Either the number of half edges is $|V(T)| - 1$ or $|V(T)|$.

If $|V(T)| = |T|$, then it contains exactly one cycle and from every edge, exactly one half is chosen. Then it is easy to see that from the unique cycle one can choose every second half-edge as this corresponds to one of the two orientations of the cycle if we regard a half-edge configuration as a partial orientation. Every other half-edge is uniquely determined. So such a component corresponds to exactly two half-edge configurations.

If $|V(T)| - 1 = |T|$, then it contains no cycle. This can happen in two different ways. In the first case one of the vertices is not covered by a half-edge, equivalently in the corresponding partial orientation its out-degree is 0. Then the position of the half-edges is uniquely determined: they are oriented toward this vertex. In the second case two halves of some edge are chosen and every other half-edge is oriented toward this edge. In the first case, the contribution is $|V(T)|$ since we can choose the sink vertex in $|V(T)|$ ways and it determines the orientation. In the second case, the contribution is $-|T|$ since we can choose the bidirected edge in $|T|$ ways and such a configuration has weight -1. So altogether we get that such a component contributes $|V(T)| - |T| = 1$ to the sum.

Hence a set A contributes to the sum $M_G(z, 1, -1)$ if and only if it is a pseudo-forest and its contribution is $2^{c(A)} z^{|E|-|A|}$. Hence

$$M_G(z,1,-1) = \sum_{A \in \mathcal{PF}(G)} 2^{c(A)} z^{|E|-|A|} = z^{|E|-n} \sum_{k=0}^{n} \left(\sum_{\substack{A \in \mathcal{PF}(G) \\ |A|=k}} 2^{c(A)} \right) z^{n-k}.$$

□

By comparing the previous two results we get the following corollary.

Corollary 13.2.5. *We have*

$$R_G(z+1) = \sum_{k=0}^{n} \left(\sum_{\substack{A \in \mathcal{PF}(G) \\ |A|=k}} 2^{c(A)} \right) z^{n-k}.$$

Now we are ready to prove Theorem 13.2.1.

13.3 From pseudo-forests to forests

For a graph G on n vertices let us introduce the polynomial

$$F_G(z) = \sum_{k=0}^{n} f_k(G) z^{n-k},$$

where $f_k(G)$ denotes the number of spanning forests of G with exactly k edges. It turns out that if $T_G(x, y)$ denotes the Tutte polynomial of the graph G, then

$F_G(z) = z^{k(G)} T_G(z+1, 1)$. Indeed, in the definition of the Tutte polynomial,

$$T_G(x, y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)} (y-1)^{k(A)+|A|-v(G)},$$

the exponent of $y-1$ is 0 if and only if $k(A) + |A| = v(G)$, that is, A is a forest and then its contribution is $z^{v(G)-|A|}$ to the sum.

Since a forest is a pseudo-forest without any cycle from Corollary 13.2.5 we immediately have $F_G(z) \leq R_G(z+1)$ for positive z. The following theorem shows that for (essentially) large girth graphs the two quantities are not too far from each other. It is known that random regular graphs contain at most $O_d(1)$ cycles of length k for every fixed k with very high probability as the random variable X_k counting the number of k-cycles has asymptotically Poisson distribution with parameter $\frac{(d-1)^k}{k}$ (see [37] and the references therein).

Lemma 13.3.1. *Let G be a graph on n vertices with average degree $\overline{d}$ such that it contains at most L cycles of length at most $g-1$. Then*

$$\left(1 + \frac{g\overline{d}}{z}\right)^{-L-n/g} R_G(z+1) \leq F_G(z) \leq R_G(z+1).$$

Proof. We have seen that the inequality $F_G(z) \leq R_G(z+1)$ is trivial, so we only need to prove the other inequality. Let F be a spanning forest of G with connected components $T_1, \dots, T_k$ including the isolated vertices, where $k = k(F)$. For each T_j let $V(T_j)$ be its induced vertex set. Let $E[V(T_j)]$ be the subset of edges of G induced by $V(T_j)$. It contains the edges of T_j, but it may contain other edges.

Furthermore, let $\ell(F)$ be the number of components that induces a graph with some cycles, that is, the number of components, where $|E[V(T_j)]| > |V(T_j)| - 1$. We may assume that the trees $T_1, \dots, T_k$ are indexed in such a way that for $j = 1, \dots, \ell(F)$ we have $|E[V(T_j)]| > |V(T_j)| - 1$. We can embed the forest F into a pseudo-forest in

$$\prod_{j=1}^{\ell(F)} (1 + (|E[V(T_j)]| - |V(T_j)| + 1))$$

ways with the same induced connected components. Indeed, at each component T_j we may add no edges, or we may add one of the $|E[V(T_j)]| - |V(T_j)| + 1$ edges. If we take into account the weights $2^{c(A)} z^{-|A|}$ for a pseudo-forest, then we get that

$$\sum_{A \in \mathcal{PF}(G)} 2^{c(A)} z^{-|A|} \leq \sum_{F \in \mathcal{F}(G)} z^{-|F|} \prod_{j=1}^{\ell(F)} \left(1 + \frac{2}{z}(|E[V(T_j)]| - |V(T_j)| + 1)\right).$$

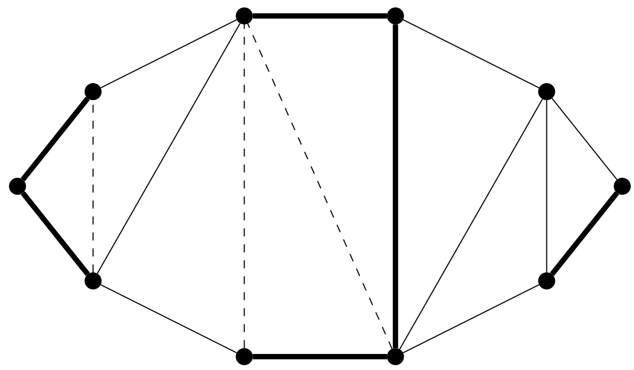


Figure 13.4: A forest F is depicted with thick edges. The dashed edges may be added to create a pseudo-forest with the same connected components, but in the middle of the picture at most one of the two dashed edges can be added.

Note that we have no equality in general since we get the same pseudo-forest from many forests. Note that if $|V(T_j)| < g$, then $E[V(T_j)]$ cannot contain a cycle with the exception of at most L components. Thus $\ell(F) \leq L + n/g$. Then by the inequality of arithmetic and geometric means we have

$$\prod_{j=1}^{\ell(F)} \left(1 + \frac{2}{z}(|E[V(T_j)]| - |V(T_j)| + 1)\right) \leq \left(\frac{1}{\ell(F)} \sum_{j=1}^{\ell(F)} \left(1 + \frac{2}{z}(|E[V(T_j)]| - |V(T_j)| + 1)\right)\right)^{\ell(F)}$$

$$\leq \left(\frac{1}{\ell(F)} \sum_{j=1}^{\ell(F)} \left(1 + \frac{2}{z}|E[V(T_j)]|\right)\right)^{\ell(F)}$$

$$\leq \left(1 + \frac{2|E|}{z\ell(F)}\right)^{\ell(F)}.$$

Since $\ell(F) \leq L + n/g$ and the function $(1 + c/t)^t$ is monotone increasing for positive t for every c we have

$$\left(1 + \frac{2|E|}{z\ell(F)}\right)^{\ell(F)} \leq \left(1 + \frac{2|E|}{z(L + n/g)}\right)^{L+n/g} \leq \left(1 + \frac{g\overline{d}}{z}\right)^{L+n/g}.$$

Hence

$$R_G(z+1) = \sum_{A \in \mathcal{PF}(G)} 2^{c(A)} z^{n-|A|} \leq \left(1 + \frac{g\overline{d}}{z}\right)^{L+n/g} F_G(z).$$

□

14. Benjamini–Schramm convergence

14.1 Introduction

It is a natural idea to consider a sequence of finite grids as a graph sequence converging to an infinite lattice. In this chapter we formalize this idea, and it will turn out that the resulting notion works in a much bigger generality than just finite subgraphs of lattices.

Definition 14.1.1 (Benjamini–Schramm convergence and estimable parameters)**.** We say that a graph sequence $(G_n)_n$ is *bounded-degree* if there is a Δ such that the maximum degree of any G_n is at most Δ.

For a finite graph G, a finite connected rooted graph α and a positive integer r, let $\mathbb{P}(G, \alpha, r)$ be the probability that the r-ball centered at a uniform random vertex of G is isomorphic to α.

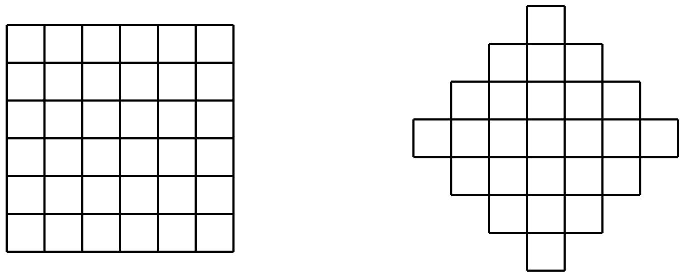
Let L be a probability distribution on (infinite) connected rooted graphs; we will call L a *random rooted graph.* For a finite connected rooted graph α and a positive integer r, let $\mathbb{P}(L, \alpha, r)$ be the probability that the r-ball centered at the root vertex is isomorphic to α, where the root is chosen from the distribution L.

We say that a bounded-degree graph sequence $(G_n)_n$ is *Benjamini–Schramm convergent* if for all finite rooted graphs α and $r > 0$, the probabilities $\mathbb{P}(G_n, \alpha, r)$ converge. Furthermore, we say that (G_n) *Benjamini-Schramm converges to L*, if for all positive integers r and finite rooted graphs α, $\mathbb{P}(G_n, \alpha, r) \to \mathbb{P}(L, \alpha, r)$.

A graph parameter $P(G)$ is estimable if for any Benjamini–Schramm convergent graph sequence the sequence $P(G_n)$ is convergent.

The Benjamini–Schramm convergence is also called *local convergence* as it primarily grasps the local structure of the graphs $(G_n)_n$.

If we take larger and larger boxes in the d-dimensional grid $\mathbb{Z}^d$, then it will converge to the rooted $\mathbb{Z}^d$, that is, the corresponding random rooted graph L is simply the distribution which takes a rooted $\mathbb{Z}^d$ with probability 1.



In the picture we depict two graph sequences both converging to $\mathbb{Z}^2$. The latter sequence consists of the so-called Aztec diamonds.

When L is a certain rooted infinite graph with probability 1, then we simply say that this rooted infinite graph is the limit without any further reference on the distribution.

There are other very natural graph sequences which are Benjamini–Schramm convergent, for instance, (G_i) is a sequence of d–regular graphs such that the girth $g(G_i) \to \infty$ (length of the shortest cycle), then it is Benjamini–Schramm convergent and we can even see its limit object: the rooted infinite d-regular tree $\mathbb{T}_d$.

There is an alternative way to look at graph parameters that are convergent whenever the graphs are Benjamini–Schramm convergent. For a vertex $v \in V(G)$ let $B_r(v)$ denote its neighborhood of radius r. Let $\mathbb{B}_r$ denote all possible r-neighborhoods, that is, the rooted graphs of radius at most r. We call a bounded graph parameter estimable, if for every $\varepsilon > 0$ there are positive integers k and r, and an "estimator" function $g : \mathbb{B}_r^k \to \mathbb{R}$ such that for every graph G and uniform, independently chosen random vertices $v_1, \dots, v_k \in V(G)$, we have

$$\mathbb{P}(|f(G) - g(B_r(v_1), \dots, B_r(v_k))| > \varepsilon) \leq \varepsilon.$$

In other words, g estimates f from a sample chosen according to the rules of sampling from a bounded degree graph. Elek [23] proved that a graph parameter is estimable if and only if it is convergent for every Benjamini–Schramm convergent graph sequence.

Theorem 14.1.2 (Elek [23]). *A bounded graph parameter f is estimable if and only if for every Benjamini–Schramm convergent graph sequence $(G_n)_n$, the sequence of numbers $(f(G_n))_n$ is convergent.*

So Benjamini–Schramm convergence coincide with a very natural setting for estimating a graph parameter.

14.2 Empirical measures

14.2.1 Matching polynomial revisited

Recall that the matching polynomial of a graph G is defined as follows. Let

$$\mu_G(z) = \sum_{k=0}^{n/2} (-1)^k m_k(G) z^{n-2k},$$

where $m_k(G)$ denotes the number of matchings of size k.

We have seen that all zeros of the matching polynomial $\mu_G(z)$ are real. Furthermore, if the largest degree Δ satisfies $\Delta \geq 2$, then all zeros lie in the interval $(-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1})$.

Let $\mu_G(z) = \prod_{i=1}^n (z-\alpha_i)$, and $s_k(G) = \sum_{i=1}^n \alpha_i^k$. We have seen that the quantities $s_k(G)$ have a combinatorial meaning. They count the so-called tree-like walks.

Note that we can introduce $s_k(\mathbb{T}_d, o)$ this way: this is simply the number of closed walks from a root vertex o of the infinite d-regular tree of length k.

Let ρ_G^m be the uniform measure supported on these roots, that is, given a function $f : \mathbb{C} \to \mathbb{C}$ we have

$$\int f(z) d\rho_G^m(z) = \frac{1}{v(G)} \sum_{i=1}^{v(G)} f(\alpha_i).$$

In particular,

$$\int z^k d\rho_G^m(z) = \frac{1}{v(G)} \sum_{i=1}^{v(G)} \alpha_i^k = \frac{s_k(G)}{v(G)}.$$

There is a measure ρ_{KM} called Kesten-McKay measure for which

$$s_k(\mathbb{T}_d, o) = \int z^k d\rho_{\mathrm{KM}}(z).$$

The Kesten-McKay measure has an explicit density function

$$\frac{d\sqrt{4(d-1)-z^2}}{2\pi(d^2-z^2)} \cdot \mathbf{1}_{(-\omega,\omega)},$$

where $\omega = 2\sqrt{d-1}$ and $\mathbf{1}_{(-\omega,\omega)}$ is the characteristic function of the interval. The following lemma is a special case of a more general theorem of Csikvári and Frenkel

[22] about Benjamini–Schramm convergent graph sequences, see also the papers [1, 2].

Lemma 14.2.1. *If $(G_n)_n$ is a sequence of d-regular graphs with $g(G_n) \to \infty$, then the measures $\rho^m_{G_n}$ converge weakly to the Kesten-McKay measure, that is, for every bounded continuous function f we have*

$$\lim_{n\to\infty} \int f(z) d\rho^m_{G_n}(z) = \int f(z) d\rho_{\mathrm{KM}}(z).$$

The proof of Lemma 14.2.1 is particularly simple since every continuous function on the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ can be approximated by a polynomial in sup norm. On the other hand, if $k < g(G)/2$, then

$$\int z^k d\rho^m_G(z) = \frac{s_k(G)}{v(G)} = s_k(\mathbb{T}_d, o) = \int z^k d\rho_{\mathrm{KM}}(z).$$

We will also need the evaluation of certain integrals along the Kesten-McKay measure.

Lemma 14.2.2. *For $0 \leq z \leq d-1$ we have*

$$z^{1/2} \exp\left(\int \ln\left(\frac{d+z-1}{\sqrt{z}} - t \right) d\rho_{\mathrm{KM}}(t) \right) = (d-1) \left(\frac{(d-1)^2}{(d-1)^2 - z} \right)^{d/2-1}.$$

For $z > d-1$ we have

$$z^{1/2} \exp\left(\int \ln\left(\frac{d+z-1}{\sqrt{z}} - t \right) d\rho_{\mathrm{KM}}(t) \right) = z \left(1 + \frac{1}{z-1} \right)^{d/2-1}.$$

The proof of Lemma 14.2.2 is actually a simple consequence of the following theorem of Mckay (see Lemma 3.5 of [39]).

Lemma 14.2.3. *Let $\omega = 2\sqrt{d-1}$, and $|\gamma| < \frac{1}{\omega}$. Then*

$$\int_{-\omega}^{\omega} \ln(1-\gamma z)\ d\rho_{\mathrm{KM}}(z) = \ln\left(\frac{1}{\eta} \left(\frac{d-1}{d-\eta} \right)^{d/2-1} \right),$$

where

$$\eta = \frac{1 - (1 - 4(d-1)\gamma^2)^{1/2}}{2(d-1)\gamma^2}.$$

Proof of Lemma 14.2.2. Let $\gamma = \frac{\sqrt{z}}{d+z-1}$. Note that $|\gamma| < \frac{1}{\omega}$ if $z \neq d-1$. The result follows by continuity if $z = d-1$. Then

$$\ln\left(\frac{d+z-1}{\sqrt{z}} - t \right) = \ln\left(\frac{d+z-1}{\sqrt{z}} \right) + \ln(1-\gamma t).$$

Note that

$$1 - 4(d-1)\gamma^2 = \frac{(d-1-z)^2}{(d+z-1)^2}$$

and so

$$\frac{1-(1-4(d-1)\gamma^2)^{1/2}}{2(d-1)\gamma^2} = \begin{cases} \frac{d+z-1}{d-1} & \text{if } \ 0 \le z \le d-1, \\ \frac{d+z-1}{z} & \text{if } \ d-1 \le z. \end{cases}$$

The rest is simple substitution. □

15. Limit Theorems I

15.1 Eulerian orientations

The main theorem of this chapter is the following one.

Theorem 15.1.1. *The parameter $\frac{1}{v(G)}\ln \varepsilon(G)$ is estimable for Eulerian graphs, that is, if $(G_n)_n$ is a Benjamini–Schramm convergent sequence of Eulerian graphs, then $\lim_{n\to\infty} \frac{1}{v(G_n)}\ln \varepsilon(G_n)$ exists.*

Proof. Recall that the following theorem connects the number of Eulerian orientations with the subgraph counting polynomial.

Theorem 15.1.2 (Borbényi and Csikvári [7])**.** *For an even number d let $\underline{s} = (s_0, s_1, \dots, s_d)$ be defined as follows.*

$$s_k = \begin{cases} \frac{\binom{d}{d/2}\binom{d/2}{k/2}}{2^{d/2}\binom{d}{k}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Then $F_G(s_0, \dots, s_d)$ counts the number of Eulerian orientations of a d–regular graph G.

The following generalization of Theorem 10.4.1 is also true.

Theorem 15.1.3 (Borbényi and Csikvári [7])**.** *Let G be an Eulerian graph. For each vertex $v \in V$ let us introduce the vector $\underline{s}^{(v)} = (s_0^{(v)}, s_1^{(v)}, \dots, s_{d(v)}^{(v)})$, where*

$$s_k^{(v)} = \begin{cases} \frac{\binom{d(v)}{d(v)/2}\binom{d(v)/2}{k/2}}{2^{d(v)/2}\binom{d(v)}{k}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Then $F_G\left(\left(s_0^{(v)}, \dots, s_{d(v)}^{(v)}\right)_{v\in V}\right)$ counts the number of Eulerian orientations of the graph G.

For a graph G let us introduce the polynomial

$$P_G(z) = F_G\left(\left(s_0^{(v)}, s_1^{(v)}, \ldots, s_{d(v)}^{(v)}\right)_{v\in V} \,\middle|\, z\right),$$

where

$$s_k^{(v)} = \begin{cases} \dfrac{\binom{d(v)}{d(v)/2}\binom{d(v)/2}{k/2}}{2^{d(v)/2}\binom{d(v)}{k}} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Example 15.1.4. For the complete graph K_5 on 5 vertices we have

$$P_{K_5}(z) = F_{K_5}\left(\frac{3}{2}, 0, \frac{1}{2}, 0, \frac{3}{2} \,\middle|\, z\right) = \frac{243}{32}z^{20} + \frac{45}{16}z^{14} + \frac{45}{32}z^{12} + \frac{3}{8}z^{10} + \frac{45}{32}z^{8} + \frac{45}{16}z^{6} + \frac{243}{32}.$$

The following picture depicts its zeros.

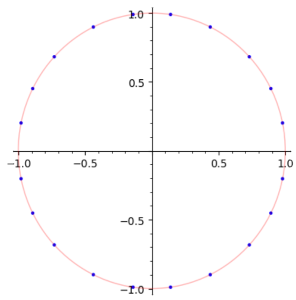


Figure 15.1: The zeros of $P_{K_5}(z)$.

By Theorem 15.1.3 we know that $P_G(1) = \varepsilon(G)$. Observe that the polynomials

$$K_v(s_0^{(v)}, \ldots, s_d^{(v)}|z) = \sum_{k=0}^{d(v)} \binom{d(v)}{k} s_k^{(v)} z^k = 2^{-d(v)/2}\binom{d(v)}{d(v)/2}(1+z^2)^{d(v)/2},$$

that is, all its zeros lie on the unit circle. By Theorem 11.2.1 it implies that the zeros of $P_G(z)$ also lie on the unit circle. If G has m edges, then the degree of the polynomial $P_G(z)$ is $2m$ and we can factorize it as follows:

$$P_G(z) = 2^{-m}\prod_{v\in V}\binom{d(v)}{d(v)/2}\prod_{i=1}^{2m}(z-\rho_i),$$

where $|\rho_i| = 1$ for $i = 1, \ldots, 2m$. Let us introduce the following measure on the complex plane:

$$\mu_G = \frac{1}{2m}\sum_{i=1}^{2m}\delta_{\rho_i},$$

where δ_s is the Dirac-measure supported on $s \in \mathbb{C}$. If k is a fixed non-negative integer, then

$$\int z^k \, d\mu_G(z) = \frac{1}{2m} \sum_{i=1}^{k} \rho_i^k.$$

If $P_G(z) = \sum_{k=0}^{2m} a_k z^k$, then the integral $\int z^k \, d\mu_G(z)$ is determined by the numbers $a_{2m}, a_{2m-1}, \dots, a_{2m-k}$ which in turn are determined by the k-neighborhood statistics of the graph G. It turns out that it implies that if $(G_n)_n$ is a Benjamini–Schramm convergent graph sequence, then the sequence $\int z^k \, d\mu_{G_n}(z)$. The precise details of this argument is given in the paper [22]. A measure sequence μ_n on $\mathbb{C}$ is convergent if for any fixed k and ℓ, the sequence $\int z^k \overline{z}^\ell \, d\mu_n(z)$ is convergent. Note that μ_{G_n} is supported on the unit circle, this is equivalent with the convergence of $\int z^k \, d\mu_{G_n}(z)$. Whence μ_{G_n} is weakly convergent.

Now let us fix some $u \neq 1$ positive real number and consider $\frac{1}{v(G_n)} \ln P_{G_n}(u)$. We have

$$\frac{1}{v(G_n)} \ln P_{G_n}(u) = \frac{1}{v(G_n)} \ln \left(2^{-e(G_n)} \prod_{v \in V} \binom{d_{G_n}(v)}{d_{G_n}(v)/2} \right) + \frac{2e(G_n)}{v(G_n)} \int \ln |u-z| \, d\mu_{G_n}(z).$$

Since μ_{G_n} are supported on the unit circle we get that $h_u(z) = \ln |u-z|$ is a continuous function on an open neighborhood of the unit circle. This gives that the sequence $\frac{1}{v(G_n)} \ln P_{G_n}(u)$ exists for $u \neq 1$ positive real number.

Let us introduce

$$p_L(u) = \lim_{n\to\infty} \frac{1}{v(G_n)} \ln P_{G_n}(u).$$

The final observation is that $p_L(u)$ is a monotone increasing continuous function. This is because $P_G(z)$ has only non-negative coefficients and so if $u_1 < u_2$, then

$$P_G(u_1) \leq P_G(u_2) \leq \left(\frac{u_2}{u_1} \right)^{2m} P_G(u_1),$$

whence

$$\frac{1}{v(G)} \ln P_G(u_1) \leq \frac{1}{v(G)} \ln P_G(u_2) \leq \frac{1}{v(G)} \ln P_G(u_1) + \frac{2e(G)}{v(G)} \ln \left(\frac{u_2}{u_1} \right).$$

This implies that

$$p_L(u_1) \leq p_L(u_2) \leq p_L(u_1) + \Delta \ln \left(\frac{u_2}{u_1} \right)$$

showing that $p_L(u)$ is a continuous and monotone increasing function. In particular, we can introduce $p_L(1) = \lim_{u\to 1} p_L(u)$ and get that

$$\lim_{n\to\infty} \frac{1}{v(G_n)} \ln P_{G_n}(1) = p_L(1),$$

that is, $\lim_{n\to\infty} \frac{1}{v(G_n)} \ln \varepsilon(G_n)$ exists. $\square$

15.2 Further remarks*

In this last section we give some remarks on the methods used in this chapter.

15.2.1 Large girth graphs

In this section we determine the limit of $\frac{1}{v(G_n)} \ln \varepsilon(G_n)$ if $(G_n)_n$ is a large girth sequence, that is, $g(G_n) \to \infty$. This limit was determined by Vergnas [53] building on the work of Schrijver [46] if $(G_n)_n$ is a sequence of d-regular graphs. Indeed, Schrijver proved the lower bound

$$\frac{1}{v(G)} \ln \varepsilon(G) \geq \ln \left(2^{-d/2} \binom{d}{d/2} \right),$$

and Vergnas proved a matching upper bound in terms of the maximal number of pairwise edge-disjoint cycles which is at most $\frac{dv(G)}{g}$ if g the length of the shortest cycle.

Here we directly rely on the proof method we did in the previous section.

Theorem 15.2.1. *Let $(G_n)_n$ be a Benjamini–Schramm convergent sequence of Eulerian graphs with maximum degree Δ and girth $g(G_n) \to \infty$. Let*

$$t_k := \lim_{n\to\infty} \frac{|\{v \mid d_{G_n}(v) = k\}|}{v(G_n)} \quad (k = 0, \dots, \Delta),$$

then

$$\lim_{n\to\infty} \frac{1}{v(G_n)} \ln \varepsilon(G_n) = \sum_{k=0}^{\Delta} t_k \ln \left(2^{-k/2} \binom{k}{k/2} \right).$$

Proof. Recall that for positive real number $u \neq 1$ we had the formula

$$\frac{1}{v(G_n)} \ln P_{G_n}(u) = \frac{1}{v(G_n)} \ln \left(2^{-e(G_n)} \prod_{v\in V} \binom{d_{G_n}(v)}{d_{G_n}(v)/2} \right) + \frac{2e(G_n)}{v(G_n)} \int \ln |u-z| \, d\mu_{G_n}(z).$$

Here the first term converges to

$$\sum_{k=0}^{\Delta} t_k \ln \left(2^{-k/2} \binom{k}{k/2} \right).$$

We only need to understand the second term. In particular, we need to understand the limit of the measures μ_{G_n}. We claim that this limit measure is the uniform measure on the unit circle. We claim that $P_G(z) = \sum_{k=0}^{2m} a_k z^k$, then $a_{2m-1} = a_{2m-2} = \cdots = a_{2m-2g+1} = 0$ if the girth is bigger than g. Since $a_k = a_{2m-k}$ by the symmetric nature of the vectors $\underline{s}^{(v)}$ we only need to see that $a_1 = \cdots = a_{2g-1} = 0$ which follows since if $A \subseteq E$ satisfies that $0 < |A| < g$, then there is a vertex v such that $d_A(v) = 1$, and then $\prod_{v \in V} s^{(v)}_{d_A(v)} = 0$. From the Newton-Waring formulas we also get that $\int z^k d\mu_G(z) = 0$ for $k = 1, \dots, 2g-1$. Since $g(G_n) \to \infty$ we get that for the limit measure μ_L we have $\int z^k \, d\mu_L(z) = 0$ for every integer $k \geq 1$. Hence μ_L is the uniform measure, and

$$\lim_{u \to 1} \int \ln(u - z) \, d\mu_L(z) = 0.$$

This completes the proof.

□

15.2.2 What goes wrong with perfect matchings?

To have a better understanding of the proof strategy used in this paper we carefully analyze another graph invariant in this section, namely, the number of perfect matchings, hereafter denoted by $\mathrm{pm}(G)$.

Clearly, if we have a graph with a lot of perfect matchings, and we delete one vertex the number of perfect matchings drops to zero. This means that we need to impose some restriction on the graph class. Note that even in the case of Eulerian orientations we needed to require that the elements of the graph sequence $(G_n)_n$ are Eulerian graphs. Unfortunately, even with the assumption that all G_n are d-regular bipartite graphs one can construct a sequence of graphs $(G_n)_n$ such that $\frac{1}{v(G_n)} \ln \mathrm{pm}(G_n)$ is not convergent [1]. Nevertheless, there is one positive result: it is convergent if G_n are not only d-regular bipartite graphs, but $g(G_n) \to \infty$ is also satisfied [1].

It is very instructive to see what goes wrong in the case of the number of perfect matchings in our proof. Suppose for simplicity that G_n are 4-regular graphs. Then $\mathrm{pm}(G) = F_G(0, 1, 0, 0, 0)$ by the definition of the subgraph counting polynomial. This would not be very useful as $F_G(0, 1, 0, 0, 0|z) = \mathrm{pm}(G) z^{v(G)}$. Fortunately, $F_G(x_0, x_1, x_2, x_3, x_4)$ takes the same value at several different places due to some invariance under "rotations", see details in [7]. In particular,

$$F_G(0,1,0,0,0) = F_G\left(1,-\frac{1}{2},0,\frac{1}{2},-1\right).$$

For this vector we have

$$K_v(z) = 1 - 2z + 2z^3 - z^4 = (1-z)^3(1+z),$$

so all zeros have absolute value 1. (There is always such a vector for $(0,1,0,\ldots,0)$ no matter what d is.) This means that

$$F_G\left(1,-\frac{1}{2},0,\frac{1}{2},-1\,\middle|\, z\right)$$

have all zeros lying on the unit circle. It even implies that the function $P_G(u) := F_G\left(1,-\frac{1}{2},0,\frac{1}{2},-1|u\right)$ is non-negative for real $u > 1$ implying that for such a u the

$$p_L(u) := \lim_{n\to\infty} \frac{1}{v(G_n)} \ln P_{G_n}(u)$$

exists. If $\mathrm{pm}(G_n) \neq 0$, then $P_{G_n}(1) \neq 0$ and we can also deduce that $P_{G_n}(u) > 0$ for $0 < u < 1$ so $p_L(u)$ exists in this case. Unfortunately, since the coefficients of $P_G(z)$ are not necessarily non-negative we cannot argue that it is monotone increasing, and that $p_L(u)$ is continuous at 1.

Though this strategy does not work in the case of perfect matchings, it is still instructive to see how gauge transformation gives us a great flexibility to choose the vectors in such a way that we can apply a Lee-Yang-type theorem.

16. Limit theorems II

16.1 Introduction

In this chapter we study the asymptotic behaviour of the Tutte polynomial of large girth regular graphs. For a graph G let $g(G)$ denote the girth of the graph, that is, the length of the shortest cycle of G. The following theorems due to McKay and Lyons are the main motivation for us.

Theorem 16.1.1 (McKay [38])**.** *Let $(G_n)_n$ be a sequence of random d-regular graphs. Then asymptotically almost surely we have*

$$\lim_{n\to\infty} \tau(G_n)^{1/v(G_n)} = \frac{(d-1)^{d-1}}{(d^2-2d)^{d/2-1}}.$$

Theorem 16.1.2 (Lyons [36])**.** *Let $(G_n)_n$ be a sequence of connected d-regular graphs such that $\lim_{n\to\infty} g(G_n) = \infty$. Then*

$$\lim_{n\to\infty} \tau(G_n)^{1/v(G_n)} = \frac{(d-1)^{d-1}}{(d^2-2d)^{d/2-1}}.$$

Lyons actually proved a more general result on Benjamini–Schramm convergent graph sequences.

Our main theorem is an analogue of these theorems for evaluations of the Tutte polynomial for a wide range of parameters.

Theorem 16.1.3. *Let $x \geq 1$ and $0 \leq y \leq 1$. Let $d \geq 2$, and let $(G_n)_n$ be a sequence of d-regular graphs such that $\lim_{n\to\infty} g(G_n) = \infty$. Then*

$$\lim_{n\to\infty} T_{G_n}(x,y)^{1/v(G_n)} = t_d(x,y),$$

where

$$t_d(x,y) = \begin{cases} (d-1)\left(\frac{(d-1)^2}{(d-1)^2-x}\right)^{d/2-1} & \text{if } x \leq d-1 \text{ and } 0 \leq y \leq 1, \\ x\left(1+\frac{1}{x-1}\right)^{d/2-1} & \text{if } x > d-1 \text{ and } 0 \leq y \leq 1. \end{cases}$$

If $(G_n)_n$ is a sequence of random d-regular graphs, then the same statement holds true asymptotically almost surely. In fact, if $L(G,g)$ denotes the number of cycles of length at most $g-1$ in a graph G, then the same conclusion holds if for every fixed g we have $\lim_{n\to\infty}\frac{L(G_n,g)}{v(G_n)}=0$.

Note that the case $x=1, y=1$ covers Theorems 16.1.1 and 16.1.2. In case of $d=2$, one needs to define $t_2(1,y)=1$. Note that for the cycle C_n on n vertices we have $T_{C_n}(x,1)=\frac{x^n-1}{x-1}$ which shows that $\lim_{n\to\infty}T_{C_n}(x,1)^{1/n}=t_2(x,1)$ even if $0\le x<1$. We believe that the statement of Theorem 16.1.3 is true for every d when $0\le x<1$ and $0\le y\le 1$ except when $x=y=0$, but our proof does not work in this case. Naturally, we can introduce the function $t_d(x,y)$ for every x and y the way it is defined in Theorem 16.1.3, but it is not a priori clear that the corresponding limit exists. The authors conjecture that the limit indeed exists whenever $x,y>0$, and if the graphs $(G_n)_n$ do not contain loops and bridges, then we can also allow $x=0$ or $y=0$. Let us mention that $t_d(x,y)$ will depend on y if y is large enough in terms of x.

For the number of spanning forests ($F(G)=T_G(2,1)$) and acyclic orientations ($a(G)=T_G(2,0)$) we immediately get the following statement.

Theorem 16.1.4. *Let $(G_n)_n$ be a sequence of d-regular graphs such that $\lim_{n\to\infty}g(G_n)=\infty$. Let $F(G)$ denote the number of spanning forests of the graph G. Similarly, let $a(G)$ denote the number of acyclic orientations of the graph G. Then*

$$\lim_{n\to\infty}F(G_n)^{1/v(G_n)}=\lim_{n\to\infty}a(G_n)^{1/v(G_n)}=\frac{(d-1)^{d-1}}{(d^2-2d-1)^{d/2-1}}.$$

If $(G_n)_n$ is a sequence of random d-regular graphs, then the same statement holds true asymptotically almost surely. In fact, if $L(G,g)$ denotes the number of cycles of length at most $g-1$ in a graph G, then the same conclusion holds if for every fixed g we have $\lim_{n\to\infty}\frac{L(G_n,g)}{v(G_n)}=0$.

We note that the special case of Theorem 16.1.4, when $d=3$, was previously proved by Borbényi, Csikvári and Luo [8].

16.2 Proof of Theorems 16.1.4 and 16.1.3 for $y=1$

In this part, we prove Theorems 16.1.4 and 16.1.3 for $y=1$.

Note that for a d-regular graph G we have

$$R_G(z) = \sum_{M \in \mathcal{M}(G)} (-z)^{|M|}(d+z-1)^{n-2|M|} = z^{n/2}\mu_G\left(\frac{d-1+z}{\sqrt{z}}\right).$$

Proof of Theorems 16.1.4 and 16.1.3. Let $x = z+1$. First we assume that $x > 1$, that is, $z > 0$. The claim will follow for $x = 1$ by continuity, see the end of the proof. Recall that $F_G(z) = z^{k(G)}T_G(z+1,1)$. By Theorem 13.3.1 we have

$$\left(1+\frac{g\overline{d}}{z}\right)^{-L-n/g} R_G(z+1) \leq F_G(z) \leq R_G(z+1)$$

if G contains at most L cycles of length at most $g-1$. Thus

$$\lim_{n\to\infty} F_{G_n}(z)^{1/v(G_n)} = \lim_{n\to\infty} R_{G_n}(z+1)^{1/v(G_n)}.$$

In case of a random d-regular graph sequence $(G_n)_n$ we use the fact that $L = O_{g,d}(1) = o(v(G_n))$ asymptotically almost surely.
Since

$$R_G(z) = \sum_{M \in \mathcal{M}(G)} (-z)^{|M|}(d+z-1)^{n-2|M|} = z^{v(G)/2}\mu_G\left(\frac{d-1+z}{\sqrt{z}}\right)$$

for d-regular graphs we have

$$\frac{1}{v(G)}\ln R_G(z+1) = \frac{1}{2}\ln(z+1) + \frac{1}{v(G)}\ln \mu_G\left(\frac{d+z}{\sqrt{z+1}}\right).$$

Let us introduce the notation $u := \frac{d+z}{\sqrt{z+1}}$. Note that

$$u = \frac{d+z}{\sqrt{z+1}} = \frac{(d-1)+z+1}{\sqrt{z+1}} = \frac{d-1}{\sqrt{z+1}} + \sqrt{z+1} \geq 2\sqrt{d-1}$$

and we have strict inequality if $z+1 \neq d-1$. Thus

$$\frac{1}{v(G)}\ln R_G(z+1) = \frac{1}{2}\ln(z+1) + \frac{1}{v(G)}\ln \mu_G(u).$$

Here

$$\frac{1}{v(G)}\ln \mu_G(u) = \frac{1}{v(G)}\sum_{i=1}^{n}\ln(u-\alpha_i) = \int \ln(u-t)\, d\rho_G^m(t).$$

If $z+1 \neq d-1$, then the function $\ln(u-t)$ is a bounded continuous function on the interval $(-2\sqrt{d-1}, 2\sqrt{d-1})$ and $\rho_{G_n}^m$ converges weakly to the Kesten-McKay measure. Hence

$$\lim_{n\to\infty}\int \ln(u-t)\, d\rho_{G_n}^m(t) = \int \ln(u-t)\, d\rho_{\mathrm{KM}}(t).$$

Putting these facts together we get that

$$\lim_{n\to\infty} F_{G_n}(z)^{1/v(G_n)} = (z+1)^{1/2}\exp\left(\int \ln\left(\frac{d+z}{\sqrt{z+1}} - t\right) d\rho_{\mathrm{KM}}(t)\right) = t_d(z+1,1),$$

where in the last step we used Lemma 14.2.2. If $z+1 = d-1$ then the claim is still true since $F_G(z)$ is a continuous, monotone increasing function just as the function $t_d(z+1,1)$. So if for every $z \neq d-2$ we have $\lim_{n\to\infty} F_{G_n}(z)^{1/v(G_n)} = t_d(z+1,1)$, then it is also true at $z = d-2$. Hence

$$\lim_{n\to\infty} ((x-1)^{k(G_n)} T_{G_n}(x,1))^{1/v(G_n)} = t_d(x,1)$$

for all $x > 1$, that is,

$$\lim_{n\to\infty} T_{G_n}(x,1)^{1/v(G_n)} = t_d(x,1).$$

Finally, for $x = 1$ we can conclude as follows. Since the coefficients of $T_G(x,1)$ are non-negative and its degree is $v(G) - k(G)$ we get that for $x > 1$ we have

$$T_G(1,1) \leq T_G(x,1) \leq x^{v(G)} T_G(1,1).$$

Hence

$$\limsup_{n\to\infty} T_{G_n}(1,1)^{1/v(G_n)} \leq \limsup_{n\to\infty} T_{G_n}(x,1)^{1/v(G_n)} = t_d(x,1)$$

and

$$\liminf_{n\to\infty} T_{G_n}(1,1)^{1/v(G_n)} \geq \liminf_{n\to\infty} (x^{-v(G_n)} T_{G_n}(x,1))^{1/v(G_n)} = \frac{t_d(x,1)}{x}.$$

Since $\lim_{x\searrow 1} t_d(x,1) = \lim_{x\searrow 1} \frac{t_d(x,1)}{x} = t_d(1,1)$ we get that

$$\lim_{n\to\infty} T_{G_n}(1,1)^{1/v(G_n)} = t_d(1,1).$$

Finally, in the particular case when $x = 2$ we get that

$$\lim_{n\to\infty} F(G_n)^{1/v(G_n)} = t_d(2,1) = (d-1)\left(\frac{(d-1)^2}{d^2-2d-1}\right)^{d/2-1}.$$

□

16.3 Proof of Theorems 16.1.4 and 16.1.3 for $0 \leq y < 1$

In this part we prove Theorems 16.1.4 and 16.1.3 for $0 \leq y < 1$. We only prove this part when $g(G_n) \to \infty$. The case when we allow a small number of short cycles is very similar.

The plan is that we reduce this case to the case $y = 1$. Recall that Theorem 6.4.5 claims that for a graph G with n vertices, m edges and the length of the shortest cycle is g, we have

$$T_G(x,0) \geq T_G(x,1)\left(1 - \frac{1}{g}\right)^{m-n+1}.$$

It shows that

$$\frac{1}{v(G)}\ln T_G(x,1) \geq \frac{1}{v(G)}\ln T_G(x,0) \geq \frac{1}{v(G)}\ln T_G(x,1) + \left(\frac{d}{2} - 1 + \frac{1}{v(G)}\right)\ln\left(1 - \frac{1}{g}\right).$$

Since for $g(G_n) \to \infty$ we get that

$$\lim_{n\to\infty} \frac{1}{v(G_n)}\ln T_{G_n}(x,1) = \lim_{n\to\infty} \frac{1}{v(G_n)}\ln T_{G_n}(x,0).$$

Since for $0 < y < 1$ we also have $T_G(x,0) \leq T_G(x,y) \leq T_G(x,1)$ by the non-negativity of the coefficients of the Tutte polynomial we also get that $\lim_{n\to\infty} \frac{1}{v(G_n)}\ln T_{G_n}(x,y)$ has the same limit.

17. Limit theorems III

17.1 Introduction

The main theorem of this chapter is the following.

Theorem 17.1.1. *Let $(G_n)_n$ be a sequence of d-regular graphs such that $\lim_{n\to\infty} g(G_n) = \infty$. Then the limit*

$$\lim_{n\to\infty} Z_{G_n}(q,w)^{1/v(G_n)} = \Phi_{d,q,w}$$

exists for $q \geq 2$ and $w \geq 0$. The quantity $\Phi_{d,q,w}$ can be computed as follows. Let

$$\Phi_{d,q,w}(t) := \left(\sqrt{1+\frac{w}{q}}\cos(t) + \sqrt{\frac{(q-1)w}{q}}\sin(t)\right)^d + (q-1)\left(\sqrt{1+\frac{w}{q}}\cos(t) - \sqrt{\frac{w}{q(q-1)}}\sin(t)\right)^d,$$

then $\Phi_{d,q,w} := \max_{t\in[-\pi,\pi]} \Phi_{d,q,w}(t)$.

We will not give the entire proof of this theorem, but we will give a detailed description of the ideas used to prove the statement.

17.2 Underlying ideas of the proof

In this section we collected the various steps of the proof of Theorem 17.1.1

17.2.1 Approximation

The first step of the proof is the combinatorial approximation introduced in Chapter 12. Recall that the so-called rank 2 approximation is defined as

$$Z_G^{(2)}(q,w) = \sum_{S\subseteq V} (1+w)^{e(S)}(q-1)^{v(G)-|S|}\left(1+\frac{w}{q-1}\right)^{e(G-S)}.$$

We have seen that for the matrix M_2' and $\underline{\nu}_2$

$$M_2' = \begin{pmatrix} 1+w & 1 \\ 1 & 1+\frac{w}{q-1} \end{pmatrix} \quad \text{and} \quad \underline{\nu}_2 = \begin{pmatrix} 1 \\ q-1 \end{pmatrix},$$

we have

$$Z_G^{(2)}(q,w) = Z_G(M_2', \underline{\nu}_2).$$

By Theorem 12.3.3 we have that for a graph G on n vertices with $L = L(G,g)$ cycles of length at most $g-1$, and $q \geq 2$ we have

$$Z_G^{(2)}(q,w) \leq Z_G(q,w) \leq q^{n/g+L} Z_G^{(2)}(q,w).$$

This shows that if G_n is a graph sequence Benjamini–Schramm convergent to the infinite d-regular tree $\mathbb{T}_d$ we have

$$\lim_{n\to\infty} Z_{G_n}(q,w)^{1/v(G_n)} = \lim_{n\to\infty} Z_{G_n}^{(2)}(q,w)^{1/v(G_n)}$$

if one of the limit exists.

17.2.2 Subgraph counting polynomial

In the next step of the proof we study $\lim_{n\to\infty} Z_{G_n}^{(2)}(q,w)^{1/v(G_n)}$. Here we use that $Z_{G_n}^{(2)}(q,w)$ can be encoded via the subgraph counting polynomial.

We can apply the argument at the end of Chapter 9 to $N = M_2'$, $\underline{\mu} = \underline{\nu}_2$ with the following vectors.

$$\underline{a} = \begin{pmatrix} \sqrt{1+\frac{w}{q}} \\ \sqrt{1+\frac{w}{q}} \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} \sqrt{\frac{(q-1)w}{q}} \\ -\sqrt{\frac{w}{q(q-1)}} \end{pmatrix}.$$

One can check that $M_2' = \underline{a}\underline{a}^T + \underline{b}\underline{b}^T$ indeed holds true. We can again introduce the vectors $\underline{a}(t), \underline{b}(t)$ giving rise to a vector $\underline{v}(t) = (r_0(t), \dots, r_d(t))$ such that

$$F_G(\underline{v}(t)) = Z_G(M_2', \underline{\nu}_2) = Z_G^{(2)}(q,w).$$

In this case

$$r_j(t) = \sum_{k=1}^{2} \mu_k a(t)_k^{d-j} b(t)_k^j$$

$$= \left(\sqrt{1+\frac{w}{q}}\cos(t)+\sqrt{\frac{(q-1)w}{q}}\sin(t)\right)^{d-j}\left(-\sqrt{1+\frac{w}{q}}\sin(t)+\sqrt{\frac{(q-1)w}{q}}\cos(t)\right)^{j}$$
$$+ (q-1)\left(\sqrt{1+\frac{w}{q}}\cos(t)-\sqrt{\frac{w}{q(q-1)}}\sin(t)\right)^{d-j}\left(-\sqrt{1+\frac{w}{q}}\sin(t)-\sqrt{\frac{w}{q(q-1)}}\cos(t)\right)^{j}.$$

In particular,

$$r_0(t) = \left(\sqrt{1+\frac{w}{q}}\cos(t)+\sqrt{\frac{(q-1)w}{q}}\sin(t)\right)^{d}+(q-1)\left(\sqrt{1+\frac{w}{q}}\cos(t)-\sqrt{\frac{w}{q(q-1)}}\sin(t)\right)^{d}.$$

In other words, $r_0(t) = \Phi_{d,q,w}(t)$.

17.2.3 Lower bound

In the third step of the proof one can show that for any non-negative positive definite matrix N, and in particular to M_2' it is true that the value t_0 that maximizes $r_0(t)$ we have $r_1(t_0) = 0$, and $r_{2j}(t_0) \geq 0$ for all $j \leq d/2$, and $r_{2j+1}(t_0) \geq 0$ for all $j \leq (d-1)/2$ or $r_{2j+1}(t_0) \leq 0$ for all $j \leq (d-1)/2$. Since any subgraph has an even number of odd degree vertex this immediately implies that

$$Z_G^{(2)}(q,w) = F_G(r_0(t_0),\dots,r_d(t_0)) \geq r_0(t_0)^{v(G)} = \Phi_{d,q,w}^{v(G)}.$$

17.2.4 Lee-Yang theorem again and convergence

The fourth step of the proof is to show that there exists a t_1 such that the polynomial

$$P_G(z) := F_G(r_0(t_1), r_1(t_1)z, r_2(t_1)z^2, \dots, r_d(t_1)z^d)$$

has all zeros lying on a circle centered at 0. In order to prove it one needs to check the condition of Theorem 11.2.1, that is, one needs to prove that for some t_1 all zeros of

$$P_{d,t_1}(z) := \sum_{k=0}^{d}\binom{d}{k} r_k(t_1) z^k$$

lie on a circle centered at 0. It turns out that for any t the zeros of $P_{d,t}(z)$ lie on a circle, but one needs some extra work to show that there is a t_1 for which the center of this circle is at 0. It turns out that the radius of the circle is almost never 1, so we can actually argue that the sequence $\frac{1}{v(G_n)}\ln P_{G_n}(1)$ –which is equal to $\frac{1}{v(G_n)}\ln Z^{(2)}_{G_n}(q,w)$– is converging for a Benjamini–Schramm convergent graph sequence of d–regular graphs G_n. The proof of this fact works the same way as for the number of Eulerian orientations. In the rare case when the radius of circle is 1 one can use continuity argument.

17.2.5 The value of the limit

Once we know that the limit exists for any Benjamini–Schramm convergent graph sequence of d–regular graphs G_n we need a last step to prove that for a graph sequence G_n converging to $\mathbb{T}_d$ the limit of $\frac{1}{v(G_n)}\ln Z^{(2)}_{G_n}(q,w)$ is $\ln \Phi_{d,q,w}$. Here one can use that for random d-regular graphs on n vertices one can show that

$$\mathbb{E}Z^{(2)}_G(q,w) = \mathrm{poly}(n)\Phi^n_{d,q,w}.$$

(Note that one needs to use the so-called configuration model of random d-regular graphs. This model is exceptionally suitable to compute such kind of expected values.) In general, it is easy to compute $\mathbb{E}Z_G(N,\underline{\mu})$, but it needs some work to show that in the obtained formula the exponential growth constant is indeed $\Phi_{d,q,w}$. Finally, taking a sequence of random d-regular graphs in general it can occur that the limits

$$\phi_1 := \lim_{n\to\infty}\frac{1}{n}\ln \mathbb{E}Z_G(N,\underline{\mu}) \quad \text{and} \quad \phi_2 := \lim_{n\to\infty}\frac{1}{n}\mathbb{E}(\ln Z_G(N,\underline{\mu}))$$

are different, but it is always true that $\phi_1 \leq \phi_2$. But since we know that $\phi_1 = \ln \Phi_{d,q,w}$ for $N = M_2'$ and $\underline{\mu} = \underline{\nu}$ and $\phi_2 \geq \ln \Phi_{d,q,w}$ since $Z^{(2)}_G(q,w) \geq \Phi^{v(G)}_{d,q,w}$ from the third step we immediately get that $\phi_1 = \phi_2 = \ln \Phi_{d,q,w}$. This finishes the proof.

18. Problems and conjectures

In this chapter I give various problems and conjectures. The numbers behind the conjectures have the following meaning:
2. Possibly doable.
3. I can imagine that there is some simple solution, and is very likely publishable.
4. Probably very hard, clearly publishable.
5. Famous open problem, publishable in a strong journal.
Note that since I do not know the solution for any problem, I might be very wrong about the difficulty level of a particular question or conjecture.

18.1 Permutation Tutte polynomial

Problem 18.1.1 (2)**.** Is it true that for any bipartite graph H and $b = a^2 + 2a$ we have

$$\widetilde{T}_H(b,0)\widetilde{T}_H(0,b) \geq \widetilde{T}_H(a,a)^2?$$

For the Tutte polynomial $T_G(x,y)$ Conjecture 18.1.1 was proved by Jackson and as we have seen the case $a = 1$ can be proved by FKG-inequality. I can prove this conjecure with $b = 3a^2$ for $a \geq 1$ instead of $b = a^2 + 2a$. In some sense the growth $b \sim a^2$ is necessary.

Problem 18.1.2 (3)**.** What is the smallest β for which

$$\widetilde{T}_H(\beta,0)\widetilde{T}_H(0,\beta) \geq \widetilde{T}_H(1,1)^2$$

for every bipartite graph H? (Any improvement over the current best would be great.)

18.2 Zeros

Conjecture 18.2.1 (4)**.** Let G be a graph with maximum degree d. Suppose that $T_G(\xi, 1) = 0$. Then $|\xi| \leq d - 1$.

The tightness of the above conjecture is also a problem.

Problem 18.2.2 (4)**.** Suppose that $(G_n)_n$ is a sequence of large girth graphs. Then the zeros of $T_{G_n}(x, 1)$ converge to the circle of radius $d - 1$ on the complex plane.

All these questions are partly motivated by the following conjecture of Sokal about the zeros of the chromatic polynomial.

Conjecture 18.2.3 (5)**.** Let G be graph with maximum degree Δ. Show that $\mathrm{ch}(G, \xi) \neq 0$ if $\mathrm{Re}(\xi) \geq \Delta + 1$.

18.3 Correlation

I call the following conjecture the rectangle correlation inequality.

Conjecture 18.3.1 (3)**.** Prove that for $0 < x_1 < x_2$ and $0 < y_1 < y_2$ and for every graph G we have

$$T_G(x_1, y_2)T_G(x_2, y_1) \geq T_G(x_1, y_1)T_G(x_2, y_2).$$

The above conjecture is true if $(x_2 - 1)(y_2 - 1) \geq 1$. There is a stronger version of the above conjecture.

Conjecture 18.3.2 (4)**.** Let $S_G(x_1, x_2, y_1, y_2)$ be defined by the following identity:

$$(y_2 - y_1)(x_2 - x_1)S_G(x_1, x_2, y_1, y_2) = T_G(x_1, y_2)T_G(x_2, y_1) - T_G(x_1, y_1)T_G(x_2, y_2).$$

Then all coefficients of $S_G(x_1, x_2, y_1, y_2)$ are non-negative.

18.4 Graph limits

Conjecture 18.4.1 (3)**.** Let $(G_n)_n$ be a Benjamini–Schramm convergent graph sequence. Let N a 2×2 positive definite matrix with positive entries, and $\underline{\mu} \in \mathbb{R}^2$ a positive vector. Then

$$\lim_{n \to \infty} Z_{G_n}(N, \underline{\mu})^{1/v(G_n)}$$

exists.

When $N_{11} = N_{22}$ or $\mu_1 = \mu_2$, then Conjecture 18.4.1 is known to be true. One may even conjecture that the statement is even true if $N \in \mathbb{R}^{q \times q}$ is a positive definite matrix with positive entries and $\underline{\mu} \in \mathbb{R}^q$ with positive entries.

Conjecture 18.4.2 (5)**.** Let $(G_n)_n$ be a Benjamini–Schramm convergent graph sequence. Then for any $x, y > 0$ the $\lim_{n\to\infty} T_{G_n}(x, y)^{1/v(G_n)}$ exists.

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