

TWO REMARKS ON THE ADJOINT POLYNOMIAL

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ABSTRACT. One can define the adjoint polynomial of the graph G as follows. Let $a_k(G)$ denote the number of ways one can cover all vertices of the graph G by exactly k disjoint cliques of G . Then the adjoint polynomial of G is

$$h(G, x) = \sum_{k=1}^n (-1)^{n-k} a_k(G) x^k,$$

where n denotes the number of vertices of the graph G . In this paper we show that the largest real root $\gamma(G)$ of $h(G, x)$ has the largest absolute value among the roots. We also prove that

$$\gamma(G) \leq 4(\Delta - 1),$$

where Δ denotes the largest degree of the graph G . This bound is sharp.

1. INTRODUCTION

Throughout this paper we consider graphs without loops and multiple edges. We follow the usual notations. We denote the vertex set and edge set of the graph G by $V(G)$ and $E(G)$, respectively. Let $N_G(u)$ denote the set of neighbors of the vertex u . The largest degree of the graph G is Δ . Let $G - e$ denote the graph obtained from G by deleting the edge e .

The adjoint polynomial of a graph G was introduced by R. Liu [10] and it is defined as follows. Let $a_k(G)$ denote the number of ways one can cover all vertices of the graph G by exactly k disjoint cliques of G . Clearly, $a_n(G) = 1$, $a_{n-1}(G) = e(G)$ is the number of edges. Then the adjoint polynomial of G is

$$h(G, x) = \sum_{k=1}^n (-1)^{n-k} a_k(G) x^k,$$

where n denotes the number of vertices of the graph G . We mention that in general the adjoint polynomial is defined without the alternating signs, but since the connection between the two forms is very trivial, it will not cause any confusion to work with this definition. On the other hand, it will turn out that it is much more convenient to work with this form.

Let $\gamma(G)$ denote the largest real root of the polynomial $h(G, x)$, which is known that it exists. This parameter is studied in a few papers; H. Zhao et al. determined the graphs with $\gamma(G) \in [0, 4]$ ([17]) and $\gamma(G) \in [4, 2 + \sqrt{5}]$

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([15]). (Note that in their papers $\beta(G) = -\gamma(G)$ with our notations since we have changed the signs of the coefficients to alternating signs.)

Clearly, the importance of the adjoint polynomial lies in the fact that it is strongly related to the chromatic polynomial [12]. More precisely, the chromatic polynomial of the complement of the graph G is

$$ch(\overline{G}, x) = \sum_{k=1}^n a_k(G)x(x-1)\dots(x-k+1).$$

On the other hand, it turns out that the roots of the adjoint polynomial behave much better than that of the chromatic polynomial. By checking the adjoint polynomial of small graphs, one may have the conjecture that they are all real. Unfortunately, it is not true, the first counterexamples were given by F. Brenti, G. F. Royle and D. G. Wagner [3]. They also proposed the problem that if the edge density of the graph is large enough then the adjoint polynomial has only real roots; this was again disproved by H. Zhao et al. [16]. On the other hand, it is known that the adjoint polynomials of triangle-free and comparability graphs [2, 3] have only real roots.

The adjoint polynomial just like the chromatic polynomial satisfies a certain multiplicativity property, namely

$$h(G_1 \cup G_2, x) = h(G_1, x)h(G_2, x),$$

where G_1, G_2 are graphs on distinct vertex set.

To obtain a recursive formula for the adjoint polynomial we need the following definition.

Definition 1.1. Let $e = (u, v) \in E(G)$ be an edge of the graph G . We define the graph $G * e$ as follows. We delete the vertices u and v from the graph G and replace them by a vertex w which we connect with the vertices $N_G(u) \cap N_G(v)$, where $N_G(u)$ and $N_G(v)$ denote the set of neighbors of the vertex u and v , respectively.

Now we are ready to give the recursive formula for the adjoint polynomial.

Proposition 1.2. [11] *Let $e = (u, v) \in E(G)$ be an edge of the graph G . Then*

$$h(G, x) = h(G - e, x) - h(G * e, x).$$

From Proposition 1.2 one can deduce the following theorem. This theorem will follow from our argument as well.

Theorem 1.3. [14] *The parameter $\gamma(G)$ exists. Moreover, if H is a proper subgraph of G then $\gamma(H) \leq \gamma(G)$.*

Let S_n denote the star on n vertices. Then

$$h(S_n, x) = x^n - (n-1)x^{n-1}$$

and so $\gamma(S_n) = n-1$. Since $S_{\Delta+1}$ is a subgraph of G we immediately obtain from Theorem 1.3 that

Corollary 1.4.

$$\gamma(G) \geq \Delta.$$

In this paper we prove the following two results.

Theorem 1.5. *Let G be a graph and let ρ be an arbitrary root of the adjoint polynomial $h(G, x)$. Let $\gamma(G)$ denote the largest real root of the adjoint polynomial. Then $|\rho| \leq \gamma(G)$.*

Theorem 1.6. *Let G be a graph with largest degree Δ . Let $\gamma(G)$ denote the largest real root of the adjoint polynomial of the graph G . Then*

$$\gamma(G) \leq 4(\Delta - 1).$$

This bound is sharp.

This paper is organized as follows. In the next section we prove Theorem 1.5. In the third section we prove Theorem 1.6.

2. PROOF OF THEOREM 1.5

In this section we prove Theorem 1.5. Our strategy will be the following. Let

$$h^*(G, x) = x^n h(G, \frac{1}{x})$$

and let us consider the power series

$$\frac{1}{h^*(G, x)} = \sum_{k=0}^{\infty} s_k(G) x^k.$$

We will prove that for all $k \geq 1$ we have $s_k(G) \geq 0$. The importance of this observation lies in the fact that we can use a powerful theorem from complex function theory, namely Pringsheim's theorem.

Lemma 2.1 (Pringsheim's theorem). [5] *If $f(z)$ is representable at the origin by a series expansion that has non-negative coefficients and radius of convergence R , then the point $z = R$ is a singularity of $f(z)$.*

By Pringsheim's theorem the observation $s_k(G) \geq 0$ implies that the root of $h^*(G, x)$ with smallest modulus is real. This would imply that the root of $h(G, x)$ with largest absolute value is real. In fact, we would also obtain that

$$\gamma(G) = \limsup_{k \rightarrow \infty} s_k(G)^{1/k}.$$

To prove that $s_k(G) \geq 0$ we prove the following stronger theorem.

Theorem 2.2. *Let H be a subgraph of the graph G . Let*

$$\frac{h^*(H, x)}{h^*(G, x)} = \sum_{k=0}^{\infty} s_k(H, G) x^k.$$

Then $s_k(H, G) \geq 0$. In particular, $s_k(K_1, G) = s_k(G) \geq 0$.

Proof. We prove the statement by induction on the number of edges of G . It is enough to prove the claim for $H = G - e$, since for an arbitrary spanning subgraph $H' = G - \{e_1, e_2, \dots, e_r\}$ we can use the identity

$$\frac{h^*(H', x)}{h^*(G, x)} = \frac{h^*(G - e_1, x)}{h^*(G, x)} \frac{h^*(G - \{e_1, e_2\}, x)}{h^*(G - e_1, x)} \cdots \frac{h^*(G - \{e_1, \dots, e_r\}, x)}{h^*(G - \{e_1, \dots, e_{r-1}\}, x)}.$$

By induction all terms except the first one have power series with non-negative coefficients. If we prove the statement for $\frac{h^*(G - e_1, x)}{h^*(G, x)}$ then the claim is true for all spanning subgraphs.

From this we obtain the statement for arbitrary subgraph H' since deleting some isolated vertices does not change $h^*(H', x)$. (So first, we delete the edges of $E(G) \setminus E(H)$ one by one and then delete the isolated vertices $V(G) \setminus V(H)$, where H is a subgraph of G .) Hence it is enough to prove that $s_k(G - e, G) \geq 0$ for all $k \geq 0$.

Note that we can rewrite the statement of Proposition 1.2 as

$$h^*(G, x) = h^*(G - e, x) - xh^*(G * e, x).$$

Hence

$$\begin{aligned} \frac{h^*(G - e, x)}{h^*(G, x)} &= \frac{h^*(G - e, x)}{h^*(G - e, x) - xh^*(G * e, x)} = \\ &= \frac{1}{1 - \frac{xh^*(G * e, x)}{h^*(G - e, x)}} = 1 + \frac{xh^*(G * e, x)}{h^*(G - e, x)} + \left(\frac{xh^*(G * e, x)}{h^*(G - e, x)} \right)^2 + \dots \end{aligned}$$

Observe that $G * e$ is a subgraph of $G - e$ and $|E(G - e)| < |E(G)|$, hence by induction the power series

$$f = \frac{xh^*(G * e, x)}{h^*(G - e, x)} = \sum_{k=0}^{\infty} s_k(G * e, G - e)x^{k+1}$$

and so the power series f^m ($m \geq 0$) have only non-negative coefficients. Hence the power series

$$\frac{h^*(G - e, x)}{h^*(G, x)} = \sum_{k=0}^{\infty} s_k(G - e, G)x^k$$

has only non-negative coefficients. \square

Corollary 2.3. *Let H be a subgraph of G then $s_k(H) \leq s_k(G)$. In particular, $\gamma(H) \leq \gamma(G)$.*

Proof. Since

$$\frac{1}{h^*(G, x)} = \frac{h^*(H, x)}{h^*(G, x)} \frac{1}{h^*(H, x)}$$

we immediately obtain that

$$s_k(G) = \sum_{j=0}^k s_j(H, G)s_{k-j}(H).$$

Since $s_0(H, G) = 1$ and all terms are non-negative we have $s_k(G) \geq s_k(H)$. The second claim follows from the observation

$$\gamma(G) = \limsup_{k \rightarrow \infty} s_k(G)^{1/k} \geq \limsup_{k \rightarrow \infty} s_k(H)^{1/k} = \gamma(H).$$

□

Remark 2.4. We note that we have used very little information about the adjoint polynomial in the proof of Theorem 2.2. We only needed that we have a recurrence relation where the subgraphs $(G - e, G * e)$ appearing in the recursive formula are also subgraphs of each other. In a similar manner one can prove an analogous result on the independence polynomial [4].

We also note that one can prove that the largest root of the adjoint polynomial is unique: there is no other root with the same modulus as the largest real root. We also mention that in case of a connected graph the multiplicity of the largest root of the adjoint polynomial is one. All these statements can be proved by following the argument of [4].

3. PROOF OF THEOREM 1.6

In this section we prove Theorem 1.6. We remark that if we only want to prove an inequality of type

$$\gamma(G) \leq c\Delta$$

with an appropriate constant c , then this can be done by the method of Alan Sokal [13] (it is also worth seeing [1]) since the argument applied to bound the absolute values of the roots of the chromatic polynomial works for the adjoint polynomial as well with a constant

$$c = \inf_a \frac{e^a}{a \log(1 + ae^{-a})} \approx 6.212.$$

We follow a bit different strategy which gives a sharp upper bound.

Our strategy will be the following. We will compare the adjoint polynomial with a modified version of the matching polynomial [6, 7, 8, 9]. Since our understanding of the matching polynomial is much deeper, this will enable us to transfer information from the theory of the matching polynomial to the theory of the adjoint polynomial.

We define the matching polynomial of the graph G as follows. Let $m_k(G)$ denote the number of matchings of G of size k . Then the matching polynomial of the graph G is

$$\mu(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k(G) x^{n-2k}.$$

It is known that all the roots of the matching polynomial are real and the largest one is at most $2\sqrt{\Delta - 1}$ if $\Delta \geq 2$ [6, 8]. We will use the following modified matching polynomial:

$$M(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k(G) x^{n-k}.$$

Since $x^n \mu(G, x) = M(G, x^2)$ we can deduce that the roots of this polynomial are non-negative real numbers, the largest one is at most $4(\Delta - 1)$. Let $t(G)$ denote the largest root of the modified matching polynomial $M(G, x)$. Hence $t(G) \leq 4(\Delta - 1)$.

It is easy to see that for triangle-free graphs the adjoint polynomial and the modified matching polynomial coincide. The modified matching polynomial satisfies the following recursive formula.

Proposition 3.1. *Let $e = (u, v) \in E(G)$. Then*

$$M(G, x) = M(G - e, x) - xM(G - \{u, v\}, x).$$

In what follows it will be more convenient to work with

$$M^*(G, x) = x^n M(G, \frac{1}{x}) = \sum_{k=0}^n (-1)^k m_k(G) x^k.$$

The next theorem is the straightforward analogue of Theorem 2.2 for matching polynomials.

Theorem 3.2. *Let H be a subgraph of the graph G . Let*

$$\frac{M^*(H, x)}{M^*(G, x)} = \sum_{k=0}^{\infty} r_k(H, G) x^k.$$

Then $r_k(H, G) \geq 0$.

Remark 3.3. For convenience we give the proof of Theorem 3.2 which does not differ from the proof of Theorem 2.2. We also mention that the combinatorial meaning of $r_k(G - u, G)$ ($u \in V(G)$) is known: it counts the closed tree-like walks of length k in G which start at the vertex u [6].

Proof. We prove the statement by induction on the number of edges of G . It is enough to prove the claim for $H = G - e$, since for an arbitrary spanning subgraph $H' = G - \{e_1, e_2, \dots, e_r\}$ we can use the identity

$$\frac{M^*(H', x)}{M^*(G, x)} = \frac{M^*(G - e_1, x)}{M^*(G, x)} \frac{M^*(G - \{e_1, e_2\}, x)}{M^*(G - e_1, x)} \cdots \frac{M^*(G - \{e_1, \dots, e_r\}, x)}{M^*(G - \{e_1, \dots, e_{r-1}\}, x)}.$$

By induction all terms except the first one have power series with non-negative coefficients. If we prove the statement for $\frac{M^*(G - e_1, x)}{M^*(G, x)}$ then the claim is true for all spanning subgraphs.

From this we obtain the statement for arbitrary subgraph H' since deleting some isolated vertices does not change $M^*(H', x)$. Hence it is enough to prove that $r_k(G - e, G) \geq 0$ for all $k \geq 0$.

Note that we can rewrite the statement of Proposition 3.1 as

$$M^*(G, x) = M^*(G - e, x) - xM^*(G - \{u, v\}, x).$$

Hence

$$\frac{M^*(G - e, x)}{M^*(G, x)} = \frac{M^*(G - e, x)}{M^*(G - e, x) - xM^*(G - \{u, v\}, x)} =$$

$$= \frac{1}{1 - \frac{xM^*(G - \{u, v\}, x)}{M^*(G - e, x)}} = 1 + \frac{xM^*(G - \{u, v\}, x)}{M^*(G - e, x)} + \left(\frac{xM^*(G - \{u, v\}, x)}{M^*(G - e, x)} \right)^2 + \dots$$

Observe that $G - \{u, v\}$ is a subgraph of $G - e$ and $|E(G - e)| < |E(G)|$, hence by induction the power series

$$f = \frac{xM^*(G - \{u, v\}, x)}{M^*(G - e, x)} = \sum_{k=0}^{\infty} r_k(G - \{u, v\}, G - e)x^{k+1}$$

and so the power series f^m ($m \geq 0$) have only non-negative coefficients. Hence the power series

$$\frac{M^*(G - e, x)}{M^*(G, x)} = \sum_{k=0}^{\infty} r_k(G - e, G)x^k$$

has only non-negative coefficients. \square

It will be convenient to introduce the following notation.

Definition 3.4. Let $f = \sum_{k=0}^{\infty} f_k x^k$ and $g = \sum_{k=0}^{\infty} g_k x^k$ be power series. We say that $f \gg g$ if $f_k \geq g_k$ for all $k \geq 0$.

In particular, $f \gg 1$ means that $f_0 \geq 1$ and $f_k \geq 0$ for $k \geq 1$. Note that Theorem 2.2 and Theorem 3.2 can be restated as

$$\frac{h^*(H, x)}{h^*(G, x)} \gg 1 \quad \text{and} \quad \frac{M^*(H, x)}{M^*(G, x)} \gg 1$$

if H is a subgraph of G .

The following proposition is trivial.

Proposition 3.5. Let $f, g, h \gg 0$ be power series. Assume that $f \gg g$. Then $fh \gg gh$. In particular, if $f, g \gg 1$ and $fg^{-1} \gg 1$ then $f \gg g$.

The next theorem is the key lemma for proving Theorem 1.6.

Theorem 3.6. Let H be a subgraph of G , then

$$\frac{h^*(G, x)}{M^*(G, x)} \left(\frac{h^*(H, x)}{M^*(H, x)} \right)^{-1} \gg 1.$$

In particular,

$$\frac{h^*(G, x)}{M^*(G, x)} \gg 1.$$

In particular,

$$\gamma(G) \leq t(G).$$

Proof. We prove the statement by induction on the number of edges of G . It is enough to prove that for any edge $e \in E(G)$ we have

$$\frac{h^*(G, x)}{M^*(G, x)} \left(\frac{h^*(G - e, x)}{M^*(G - e, x)} \right)^{-1} \gg 1.$$

Indeed, if H is a proper subgraph of G then for some edge e , H is a subgraph of $G - e$. By induction we have

$$\frac{h^*(G - e, x)}{M^*(G - e, x)} \left(\frac{h^*(H, x)}{M^*(H, x)} \right)^{-1} \gg 1.$$

and if we prove the statement for G and $G - e$ then we have

$$\frac{h^*(G, x)}{M^*(G, x)} \left(\frac{h^*(G - e, x)}{M^*(G - e, x)} \right)^{-1} \frac{h^*(G - e, x)}{M^*(G - e, x)} \left(\frac{h^*(H, x)}{M^*(H, x)} \right)^{-1} \gg 1.$$

Hence

$$\frac{h^*(G, x)}{M^*(G, x)} \left(\frac{h^*(H, x)}{M^*(H, x)} \right)^{-1} \gg 1.$$

Let us start to prove the statement for G and $G - e$.

$$\begin{aligned} \frac{h^*(G, x)}{M^*(G, x)} \left(\frac{h^*(G - e, x)}{M^*(G - e, x)} \right)^{-1} &= \frac{h^*(G - e, x) - xh^*(G * e, x)}{M^*(G - e, x) - xM^*(G - \{u, v\}, x)} \left(\frac{h^*(G - e, x)}{M^*(G - e, x)} \right)^{-1} = \\ &= \frac{1 - x \frac{h^*(G * e, x)}{h^*(G - e, x)}}{1 - x \frac{M^*(G - \{u, v\}, x)}{M^*(G - e, x)}}. \end{aligned}$$

Let

$$g = x \frac{h^*(G * e, x)}{h^*(G - e, x)} \quad \text{and} \quad f = x \frac{M^*(G - \{u, v\}, x)}{M^*(G - e, x)}.$$

Then

$$\frac{1 - g}{1 - f} = \frac{1 - f + f - g}{1 - f} = 1 + \frac{f - g}{1 - f} = 1 + (f - g) \sum_{k=0}^{\infty} f^k.$$

By Theorem 2.2 and Theorem 3.2 we have $f, g \gg 0$. Thus $\sum_{k=0}^{\infty} f^k \gg 0$. So we only have to prove that $f \gg g$. This is indeed true (we write here the required inequalities and we explain it later):

$$\frac{M^*(G - \{u, v\}, x)}{M^*(G - e, x)} = \frac{M^*(G * e, x)}{M^*(G - e, x)} \frac{M^*(G - \{u, v\}, x)}{M^*(G * e, x)} \gg \frac{M^*(G * e, x)}{M^*(G - e, x)} \gg \frac{h^*(G * e, x)}{h^*(G - e, x)}.$$

Here the first inequality follows from Theorem 3.2: $G * e$ is subgraph of $G - e$ and $G - \{u, v\}$ is a subgraph of $G * e$. Thus we have

$$\frac{M^*(G * e, x)}{M^*(G - e, x)} \gg 1 \gg 0 \quad \text{and} \quad \frac{M^*(G - \{u, v\}, x)}{M^*(G * e, x)} \gg 1.$$

The inequality

$$\frac{M^*(G * e, x)}{M^*(G - e, x)} \gg \frac{h^*(G * e, x)}{h^*(G - e, x)}$$

follows from the following inequalities:

$$\frac{h^*(G - e, x)}{M^*(G - e, x)} \left(\frac{h^*(G * e, x)}{M^*(G * e, x)} \right)^{-1} \gg 1 \quad \text{and} \quad \frac{h^*(G * e, x)}{h^*(G - e, x)} \gg 1.$$

Here the first inequality follows from the induction applied to the graphs $(G * e, G - e)$, the second inequality follows from Theorem 2.2. Hence we have proved that

$$\frac{h^*(G, x)}{M^*(G, x)} \left(\frac{h^*(H, x)}{M^*(H, x)} \right)^{-1} \gg 1.$$

The second statement of the theorem follows from the first one applied to G and $H = K_1$. The third inequality follows from the observation that since

$$\frac{h^*(G, x)}{M^*(G, x)} \gg 1$$

we have

$$h^*(G, x) \geq M^*(G, x) > 0$$

on the interval $[0, \frac{1}{t(G)})$ so $\frac{1}{\gamma(G)} \geq \frac{1}{t(G)}$, i. e., $\gamma(G) \leq t(G)$. \square

Remark 3.7. After checking the proof one can see that the new ingredient in the proof of Theorem 3.6 compared to the proof of Theorem 2.2 was that $G - \{u, v\}$ is a subgraph of $G * e$. This simple observation “induces an ordering on the recurrence relations”.

Proof of Theorem 1.6. The upper bound simply follows from Theorem 3.6:

$$4(\Delta - 1) \geq t(G) \geq \gamma(G).$$

Now we prove that one cannot improve on the bound $4(\Delta - 1)$. Consider the following sequence of trees $\{T_n\}$. Let $T_n = T_{n, \Delta-1}$ be the $(\Delta - 1)$ -ary tree of depth n . This is the rooted tree which has a root of degree $\Delta - 1$, all other non-leaf vertices have degree Δ and every leaves have distance n from the root. For trees the matching polynomial $\mu(T, x)$ coincides with the characteristic polynomial of the adjacency matrix of the tree. Hence for a tree T we have $t(T) = \lambda^2(T)$, where $\lambda(T)$ is the spectral radius of the tree T . Note that the spectral radius of a d -ary tree of depth n is

$$\lambda(T_{n,d}) = 2\sqrt{d} \cos \frac{\pi}{n+2}.$$

On the other hand, we have $h(T_n, x) = M(T_n, x)$ since T_n is triangle-free. Thus

$$\gamma(T_n) = t(T_n) = 4(\Delta - 1) \cos^2 \frac{\pi}{n+2}.$$

This implies that

$$\lim_{n \rightarrow \infty} \gamma(T_n) = 4(\Delta - 1).$$

\square

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