# CHROMATIC ROOTS AND LIMITS OF DENSE GRAPHS

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ABSTRACT. In this short note we observe that recent results of Abért and Hubai and of Csikvári and Frenkel about Benjamini–Schramm continuity of the holomorphic moments of the roots of the chromatic polynomial extend to the theory of dense graph sequences. We offer a number of problems and conjectures motivated by this observation.

## 1. INTRODUCTION

Recently, there has been much work on developing limit theories of discrete structures, and of graphs in particular. The best understood limit concepts are those for dense graph sequences and bounded-degree graph sequences. The former one was developed by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi [6, 15], and the later was initiated by Benjamini and Schramm [5]. The convergence notion in both these theories is based on frequencies of finite subgraphs, and it is a fundamental programme to understand what other parameters are captured in the limits (i.e., are continuous with respect to the corresponding topologies). In this short note we show that recent proofs of Abért and Hubai and of Csikvári and Frenkel about the convergence of holomorphic moments of the chromatic roots in a Benjamini–Schramm convergent sequence translate to the dense model as well. Furthermore, we conjecture that in the dense model it is actually true that we have weak convergence of the root distributions.

Let us now give the details. We assume the reader's familiarity with basics of graph limits; see Lovász's recent monograph [14]. Recall that given a graph $G$ of order $n$, its *chromatic polynomial* $P(G, x)$ (in a complex variable $x$) is defined as

$$(1.1) \qquad P(G, x) = \sum_{k=0}^{n} \mathrm{ip}(G, k) x(x-1) \dots (x-k+1) \,,$$

where $\mathrm{ip}(G, k)$ is the number of partitions of $V(G)$ into $k$ non-empty independent sets. In other words, for integer values $x$, $P(G, x)$ counts the number of proper vertex-colorings of $G$ with $x$ colors.

Next, we recall the result of Abért and Hubai [3]. Suppose that $G$ is a graph of maximum degrees at most $D$. We can associate to it the uniform probability measure $\mu_G$ on the multiset of its the roots of the chromatic polynomial $P(G, \cdot)$. The Sokal

bound [16] tells us that $\mu_G$ is supported in the disk of radius (strictly less than) $8D$. The main result of [3] then reads as follows.

**Theorem 1.1.** *Suppose that $(G_n)_n$ is a Benjamini–Schramm convergent sequence of graphs of maximum degree at most $D$. Suppose that $f : B \to \mathbb{C}$ is a holomorphic function defined on the open disk $B = B(0, 8D)$. Then the sequence*

$$\int f(z)\mathrm{d}\mu_{G_n}(z)$$

*converges.*

Note that to prove Theorem 1.1 it suffices to prove the convergence of the *holomorphic moments*

$$\int z^k \mathrm{d}\mu_{G_n}(z) \qquad (k \in \mathbb{N})$$

for a Benjamini–Schramm convergent graph sequence $(G_n)_n$, and this is indeed how the proof goes.

As was noted in [3], it is not always the case that the measures $\mu_{G_n}$ in a Benjamini–Schramm convergent graph sequence converge weakly. This can be seen from the following example.

**Example 1.2.** *Consider paths $P_n$ and cycles $C_n$ of growing order. These two sequences have the same Benjamini–Schramm limit but the weak limit of $(\mu_{P_n})_{n\to\infty}$ is concentrated on 1 whereas the weak limit of $(\mu_{C_n})_{n\to\infty}$ is the uniform measure on the unit circle with the center in 1.*

Csikvári and Frenkel [8] generalized Theorem 1.1 to a wider class of polynomials. This is discussed in Section 4.2.

Let us now turn to dense graphs. Suppose that $G$ is a graph of order $n$. Then the vertices of $G$ have arbitrary degrees between 0 and $n - 1$. The measure $\mu_G$ need not be supported in a bounded region for a such a graph $G$; the Sokal bound gives only a bound of roughly $8n$ on the modulus of the chromatic roots. This bound can probably be improved down to $n - 1$ (see Conjecture 4.1) but not more. Thus, it is natural to scale down $\mu_G$ by the factor of $n$, defining a new probability measure $\nu_G$, $\nu_G(X) := \mu_G(nX)$, where for $X \subset \mathbb{C}$ we define $nX = \{nx : x \in X\} \subset \mathbb{C}$. Now, $\nu_G$ is supported in the disk of radius 8. The main result of this note is the observation that Theorem 1.1 has a counterpart for sequence of dense graphs.

**Theorem 1.3.** *Suppose that $(G_n)_n$ is a sequence of graphs which converges in the dense model. Suppose that $f : B \to \mathbb{C}$ is a holomorphic function defined on an open disk $B = B(0, 8)$. Then the sequence*

$$\int f(z)\mathrm{d}\nu_{G_n}(z)$$

*converges.*

We will give a sketch of a proof of Theorem 1.3 in Section 2.

Note that by a standard argument from complex analysis, we can approximately count the number of colorings in a convergent graph sequence. Note that when $G$ has $n$ vertices and $\ell = Cn$, we expect $P(G, \ell)$ to be of the form $(cn)^n$ for some $c \in \mathbb{R}$.

**Theorem 1.4.** *Let $(G_n)_n$ be a sequence of graphs convergent in the dense model, where $G_n$ has order $n$. Then for each $C > 8$, the quantity*

$$\frac{\sqrt[n]{P(G_n, Cn)}}{n}$$

*converges as $n \to \infty$.*

The proof follows the same lines as Theorem 1.2 of [3]. Also, the result can be stated a bit more generally, as can again be seen in Theorem 1.2 of [3].

We believe that there exist no counterpart to Example 1.2 for dense graphs. This is the main conjecture of the present paper.

**Conjecture 1.5.** *Suppose that $(G_n)_n$ is a sequence of graphs convergent in the dense model. Then the measures $\nu_{G_n}$ converge weakly.*

In general, a graphon does not carry much information about chromatic properties of graphs which converge to it. For example, it is easy to construct a sequence of graphs such that their chromatic numbers grow almost linearly with their orders, yet converge to the constant-zero graphon. On the other hand it is easy to construct another sequence of graphs such that their chromatic numbers grow arbitrarily slowly, yet converge to the constant-one graphon. That is, in a sense, the chromatic number is not even semicontinuous with respect to the cut-distance.

An immediate consequence of Conjecture 1.5 would be that it would allow us to associate "chromatic roots" to graphons. This is perhaps the most substantial information about chromatic properties which could be reflected in the limit.

The only support for Conjecture 1.5 is a lack of counterexamples we could come up with. In particular, the Conjecture asserts that the normalized chromatic measures of Erdős–Rényi random graphs (with constant edge probability) or more generally random graphs coming from sampling from a graphon converge — and this seems to be a very weak form of the conjecture. It would be very interesting to prove this, and to describe the weak limit.

**Problem 1.6.** *What is the typical distribution of the chromatic roots of the Erdős–Rényi random graph $\mathbb{G}_{n,p}$, for a fixed $p \in (0,1)$?*

Computational restriction allowed us to run simulations only for $n \leq 10$. Such limited simulations did not hint for any limit behavior.

Last, let us remark that the measure $\nu_G$ is not trivialized by the scaling we introduced. This is stated in the following proposition.

**Proposition 1.7.** *For every $\delta > 0$ there exists $\epsilon > 0$ such that the following holds. Suppose that $G$ is a graph of order $n$ with at least $\delta n^2$ edges. Then at least $\epsilon n$ of the chromatic roots of $G$ have modulus at least $\epsilon n$.*

## 2. Proof of Theorem 1.3

It is only needed to observe that the argument in [3] is valid even in the dense model. More precisely, in Theorem 3.4 of [3] the following is proven. The symbol $\hom(T, H)$ stands for the number of homomorphisms from a graph $T$ to a graph $H$.

**Theorem 2.1** (Theorem 3.4 of [3])**.** *Let $H$ be a graph, and for $k \in \mathbb{N}$ let*

$$p_k = |V(H)| \int z^k \mathrm{d}\mu_{G_n}(z).$$

*Then*

$$p_k = \sum_T (-1)^{k-1} k c_k(T) \hom(T, H) , \tag{2.1}$$

*where $c_k(T)$ are constants, and the summation ranges over connected graphs $T$ of order at most $k + 1$.*

With this result, the proof of Theorem 1.3 is straightforward. Let us write $p_{k,n}$ for the number $p_k$ from the previous theorem associated to the graph $G_n$. As was remarked earlier, it suffices to prove the theorem for $f(z) = z^k$, $k \in \mathbb{N}$. For simplicity, let's assume that the graph $G_n$ has $n$ vertices. We have

$$\int z^k \mathrm{d}\nu_{G_n}(z) = \frac{1}{n^k} \int z^k \mathrm{d}\mu_{G_n}(z) = \frac{p_{k,n}}{n^{k+1}} .$$

The sequence $(G_n)$ is convergent. In particular, for every graph $T$ of order at most $k + 1$ the quantity

$$\frac{\hom(T, G_n)}{n^{k+1}}$$

converges. Observe that the right-hand side of (2.1) (for a fixed number $k$) contains only a bounded number of summands. Consequently,

$$\frac{p_{k,n}}{n^{k+1}}$$

converges as $n \to \infty$, finishing the proof.

## 3. Proof of Proposition 1.7

It is well known, and easy to see from the formula (1.1), that the sum of the chromatic roots of $G$ is the number of edges in $G$. By the assumption of the proposition, this is at least $\delta n^2$. Also, recall that the chromatic roots are contained in the disk of radius $8n$. Thus, for

$$\delta n^2 \leq \sum_{x \text{ chr.root}} x$$

to hold, we must have that at least $\frac{\delta}{9} n$ roots $x$ of the chromatic polynomial with $\Re(x) \in \left[\frac{\delta}{9} n, 8n\right]$.

## 4. Remarks and conjectures

4.1. **Variants of Sokal's bound.** Recall that the bound asserts that if a graph has maximum degree $\Delta$, then all the chromatic roots lie in the disk of radius $r = 8\Delta$. The value $8\Delta$ is not optimal; Sokal himself actually gives $7.96\ldots \times \Delta$. On the other hand the complete bipartite graph $K_{\Delta,\Delta}$ shows [17] that one cannot go below $r = 1.59\ldots\Delta$.[1] Here, we suggest to bound the moduli of the chromatic roots by the order instead of the maximum degree.

**Conjecture 4.1.** *Every graph $G$ of order $n$ has all the chromatic roots of modulus at most $n - 1$.*

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[1]Let us note that this has not been proven rigorously.

If true, complete graphs would be the extremal graphs for the problem.[2] Note that Conjecture 4.1 is known to be true for real zeros. Indeed, if $x > n-1$ is real then each summand in (1.1) is non-negative, and the summand for $k = n$ is strictly positive, yielding $P(G, x) > 0$. Secondly, we claim that if $x$ is negative then it is not a root of $P(G, \cdot)$. Indeed, it is well-known (see e.g. [9, Corollary 2.3.1]) that the coefficients of $P(G, \cdot)$ alternate in sign. The value $P(G, x)$ is then a sum of terms with the same sign, and in particular, non-zero.

By enumerating all graphs of a given order on a computer, we have verified Conjecture 4.1 for $n \leq 10$.

Our next problem can be seen as an extension of Sokal's bound, but is also connected to Conjecture 1.5 as we show below.

**Problem 4.2.** *Suppose that $G$ is a graph and $G'$ is obtained from $G$ by adding edges in such a way that the degree at each vertex increases by at most $\Delta$. Is it true that the chromatic roots move by at most $c\Delta$, for some absolute constant $c$? (By "moving" we mean that there is a bijection $\pi$ from the multiset of the chromatic roots of $G$ to the multiset of the chromatic roots of $G'$ so that $|x - \pi(x)| \leq c\Delta$ for each chromatic root $x$ of $G$.)*

Note that to answer Problem 4.2 in positive, it would suffice to prove the case $\Delta = 1$.

Suppose that $G_1$ and $G_2$ are two $n$-vertex graphs with edit-distance at most $\epsilon n^2$. That means that after a suitable vertex identification of $V(G_1)$ and $V(G_2)$ the graph $G$ on the same vertices whose edges are the common edges of $G_1$ and $G_2$ has the property that for at most $2\sqrt{\epsilon}n$ vertices we have $\deg_G(v) \leq \deg_{G_1}(v) + \sqrt{\epsilon}n$ and $\deg_G(v) \leq \deg_{G_2}(v) + \sqrt{\epsilon}n$. For the sake of drawing the link to Conjecture 1.5, let us assume that there are no such exceptional vertices. Then a positive solution to Problem 4.2 would give that the chromatic measure $\nu_{G_1}$ and $\nu_{G_2}$ are close in the weak* topology. In particular, a positive answer to Problem 4.2 would provide a support for Conjecture 1.5 when the topology generated by the cut-distance is replaced by the stronger $L^1$-metric.

4.2. **Matching polynomial.** As mentioned before, Theorem 1.1 has been [8] extended to a large class of "multiplicative graph polynomials of bounded exponential type". (In particular, this includes univariate polynomials derived from the Tutte polynomial, and a modified version of the matching polynomial. For the matching polynomial, the behavior of the root distribution in Benjamini–Schramm convergent graph sequences was discussed in [1, 2].)

The proof in [8] of this more general statement translates verbatim to the dense setting as well,[3] thus giving Theorem 1.3 for multiplicative graph polynomials of bounded exponential type. Problem 1.6 can be asked for these alternative graph polynomials as well.

The case of the matching polynomial is particularly simple. We recall the definition. Let $G = (V, E)$ be a finite graph on $v(G) = n$ vertices. Let $m_k(G)$ be the number of $k$-matchings. Note that $m_0(G) = 1$ and $m_k(G) = 0$ for $k > \lfloor n/2 \rfloor$. The *matching*

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[2]Recall that the roots of the chromatic polynomial of a complete graph $K_n$ are $\{0, 1, \ldots, n-1\}$.

[3]Let us remark that the proof is quite different from the original proof by Abért and Hubai.

*polynomial* $\mu(G,x)$ in one variable $x$ is defined as

$$\mu(G,x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k(G) x^{n-2k}.$$

A well-known result of Heilmann and Lieb [12] asserts that the roots of the matching polynomial are all real. It is easy to see that the matching polynomial is multiplicative (w.r.t. disjoint union) and the coefficient of $x^{n-i}$ is a linear combination of subgraph counts. Thus, the version of the main result of [8] for dense graphs applies. Since convergence of (holomorphic) moments is equivalent to weak convergence for distributions supported on the real line, we get the following. If $(G_n)$ is a sequence of graphs converging in the dense model, consider the uniform distribution $\pi_n$ on the roots of the matching polynomial $\mu(G_n,x)$. Then $\pi_n$ scaled down by a factor of $v(G_n)$ converges weakly. Let us explain why this corollary is trifling. Indeed, the full Heilmann–Lieb theorem asserts that if $G$ is a graph of maximum degree $D$, and $G$ is not a matching, than the roots of the matching polynomial $\mu(G,x)$ lie in the interval $\left[-2\sqrt{D-1}, 2\sqrt{D-1}\right]$, and in particular in $\left[-2\sqrt{v(G)-2}, 2\sqrt{v(G)-2}\right]$. In other words, the distribution $\pi_n$ scaled down by a factor of $v(G_n)$ converges to the Dirac measure at 0. So, the rescaling suggested by the Heilmann–Lieb theorem is by a factor of $\sqrt{v(G_n)}$. To get the right statement, we need to introduce the *modified matching polynomial*. This is a polynomial in one variable $x$ defined by

$$M(G,x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k m_k(G) x^{n-k}.$$

The matching polynomial and its modified version encode the same information. Indeed, we have $\mu(G,x) = x^{-n} M(G,x^2)$. We can factor $M(G,x)$ as

$$(4.1) \qquad\qquad M(G,x) = x^{\lceil n/2 \rceil} \prod_{i=1}^{\lfloor n/2 \rfloor} (x - \gamma_i(G)) .$$

Then the real numbers

$$\left(\pm\sqrt{\gamma_i(G)}\right)_{i=1}^{\lfloor n/2 \rfloor},$$

together with an extra zero if $n$ is odd, are the roots of $\mu(G,x)$.

It can be easily checked directly from [8, Definitions 1.3, 1.4] that $M(G,x)$ is a graph polynomial of bounded exponential type. So, it is the *modified* matching polynomial that we want to apply the main result of [8] to. We thus readily obtain a version of Corollary (1.5) with the right scaling.

**Theorem 4.3.** *Suppose that $(G_n)_n$ is a sequence of graphs convergent in the dense model. Let $\pi_n$ be the uniform probability measure on roots of the matching polynomial $\mu(G_n,x)$. Then the rescaled measures*

$$\lambda_n(X) := \pi_n \left( \sqrt{v(G_n)} X \right)$$

*converge weakly.*

In particular, this allows us to associate "matching measure" to graphons (cf. text below Conjecture 1.5).

In the rest of this section we answer the counterpart of Problem 1.6 for the matching polynomial. This was done independently, and prior to the current manuscript being

publicly available, in [7]. Our proof relates the roots of the matching polynomial of $G_n$ to those of the complete graphs $K_n$. To compare, the proof in [7] goes via counting "tree-like walks", a concept introduced in [10].

We can now state the main result of [7].

**Theorem 4.4.** *Let $p \in (0, 1)$, and let $(G_n)_n$ be a sequence of Erdős–Rényi random graphs $G_n \sim \mathbb{G}_{n,p}$. Let $\pi_n$ be the uniform probability distribution on the roots of the matching polynomial of $G_n$. Then almost surely, the measures $\lambda_n(X) := \pi_n\left(\sqrt{n}X\right)$ converge weakly to the semicircle distribution $SC_p$ whose density function is*

$$\rho_p(x) := \frac{1}{2\pi}\sqrt{4 - \frac{x^2}{p}}, \quad -2p \le x \le 2p.$$

In combination with Theorem 4.3, this determines the limit of matching measures for an arbitrary sequence of quasirandom (in the sense of Chung–Graham–Wilson) graphs. Also, note that we present a proof only for $p$ fixed, but the same technique works also for

$$p = \Omega\left(\frac{\log^{\text{const}} n}{n}\right).$$

*Proof.* Since all the roots of the matching polynomial are real, the convergence of the holomorphic moments $\int z^k \mathrm{d}\lambda_n(z)$, $k \in \mathbb{N}$ readily implies convergence in distribution. Let us thus argue that for each $k \in \mathbb{N}$, almost surely we have

$$\int z^k \mathrm{d}\lambda_n(z) \to \int z^k \mathrm{d}SC_p(z).$$

For each fixed $k = 0, 1, 2, \ldots$, and for the random graphs $G_n$, we asymptotically almost surely have

$$(4.2) \qquad \frac{m_k(G_n)}{m_k(K_n)} = (1 + o(1))p^k,$$

as each set of $k$ pairs of vertices has probability $p^k$ of being entirely included as edges of $G_n$, and this quantity is concentrated around the expectation. For details on how to prove such a result, see Chapter 4 of [4]. Since the $m_i(G)$ are elementary symmetric polynomials of the roots $\gamma_i(G)$ (c.f. (4.1)), the Newton identities give that for each fixed $k$,

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \gamma_i(G)^k = P_k(m_1(G), \ldots, m_k(G))$$

for some multivariate polynomial $P_k$. It follows from the Newton identities that the polynomial $P_k$ has the property that

$$P_k(a_1 t, \ldots, a_k t^k) = t^k P(a_1, \ldots, a_k).$$

Putting this together with (4.2), we get that

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \gamma_i(G)^k = (1 + o(1))p^k P_k(m_1(K_n), \ldots, m_k(K_n)).$$

We conclude that

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \gamma_i(G)^k = (1 + o(1))p^k \sum_{i=1}^{\lfloor n/2 \rfloor} \gamma_i(K_n)^k.$$

By a classical result of Heilmann and Lieb [12] (see (3.15) there) the matching poly-nomials of complete graphs are the Hermite polynomials. The distribution of zeros of the Hermite polynomial of degree $n$ scaled down by $\sqrt{2n}$ converges to the semicircle distribution $SC_1$, see for instance [13]. Hence almost surely the measures $\lambda_n$ converge weakly to the semicircle distribution $SC_p$. (Note that the zeros of the matching poly-nomial are supported on $\pm\sqrt{\gamma_i}$ so we have to rescale the semicircle distribution only by a factor of $\sqrt{p}$.) $\qquad\square$

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