CO-ADJOINT POLYNOMIAL

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Abstract. In this note we study a certain graph polynomial arising from a special recursion. This recursion is a member of a family of four recursions where the other three recursions belong to the chromatic polynomial, the modified matching polynomial, and the adjoint polynomial, respectively. The four polynomials share many common properties, for instance all of them are of exponential type, i. e., they satisfy the identity

$$\sum_{S\subseteq V(G)} f(G[S],x)f(G[V\setminus S],y)=f(G,x+y)$$

for every graph G.

It turns out that the new graph polynomial is a specialization of the Tutte polynomial.

1. Introduction

Throughout this paper every graphs are simple. Let us consider the following recursion of a graph polynomial. Let $e=(u,v)\in E(G)$ and assume that $P(G,x)$ satisfies the following recursion formula

$$P(G,x)=P(G-e,x)-P(G\Delta e,x),$$

where $G\Delta e$ denotes the following graph. We delete the vertices u and v from G and replace it with a vetrex w which we connect to those vertices of $V(G)-\{u,v\}$ which were adjacent to exactly one of u and v in G. In other words, we connect w with the symmetric difference of $N(u)\setminus\{v\}$ and $N(v)\setminus\{u\}$. The Δ refers to the symmetric difference in the recursive formula. Let $\overline{K_n}$ be the empty graph on n vertices and let $P(\overline{K_n},x)=x^n$. This completely determines the graph polynomial $P(G,x)$ by induction on the number of edges. On the other hand, it is not clear at all that this graph polynomial exists since we can determine $P(G,x)$ by choosing edges in different order and we have to get the same polynomial. It will turn out that this polynomial indeed exists and it is a specialization of the Tutte polynomial. Let us call this graph polynomial co-adjoint polynomial until we don't find a better name.

What motivates this recursive formula of $P(G,x)$? Let us consider the following three graph polynomials.

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1. Let $M(G,x) = \sum_{k=0}^{n}(-1)^k m_k(G)x^{n-k}$ be the (modified) matching polynomial [2, 3, 5] where $m_k(G)$ denotes the number of matchings of size k with the convention $m_0(G) = 1$. Then $M(G,x)$ satisfies the following recursive formula: let $e = (u,v) \in E(G)$ then

$$M(G,x) = M(G-e,x) - M(G\emptyset e,x) = M(G-e,x) - xM(G-\{u,v\},x),$$

where $G\emptyset e$ denotes the following graph. We delete the vertices u,v from G and replace it with a vertex w which we do not connect with anything.

2. Let $ch(G,x)$ be the chromatic polynomial [9]. It is known that it satisfies the following recursive formula. Let $e = (u,v) \in E(G)$ then

$$ch(G,x) = ch(G-e,x) - ch(G\cup e,x) = ch(G-e,x) - ch(G/e,x),$$

where $G/e = G\cup e$ denotes the following graph. We delete the vertices u,v from G and replace it with a vertex w which we connect with the *union* of $N(u)\setminus\{v\}$ and $N(v)\setminus\{u\}$.

3. Let $h(G,x)$ be the following graph polynomial. Let $a_k(G)$ be the number of ways one can cover the vertex set of the graph G with exactly k disjoint cliques of G. Let

$$h(G,x) = \sum_{k=1}^{n}(-1)^{n-k}a_k(G)x^k.$$

The graph polynomial $h(G,x)$ is called adjoint polynomial [7, 8] (most often without alternating signs of the coefficients). Then $h(G,x)$ satisfies the following recursive formula. Let $e = (u,v) \in E(G)$ then

$$h(G,x) = h(G-e,x) - h(G\cap e,x),$$

where $G\cap e$ denotes the following graph. We delete the vertices u,v from G and replace it with a vertex w which we connect with the *intersection* of $N(u)\setminus\{v\}$ and $N(v)\setminus\{u\}$.

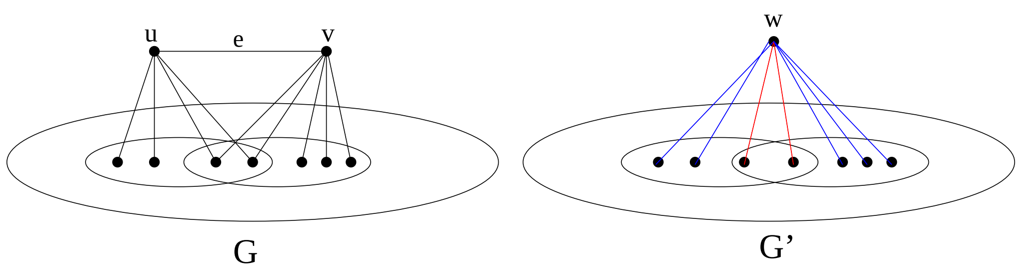

FIGURE 1. $f(G,x) = f(G-e,x) - f(G',x)$, where in G' we consider the red, blue, all or no edges according to f is the adjoint, co-adjoint, chromatic or matching polynomial, respectively.

Now it is clear that the co-adjoint polynomial is the natural fourth member of this family.

This paper is organized as follows. In the next section we prove that the co-adjoint polynomial is a specialization of the Tutte polynomial. In the third section we concern with the corollaries of this result. In the last section we study the co-adjoint polynomials of complete graphs and balanced complete bipartite graphs.

2. Specialization of the Tutte polynomial

The Tutte polynomial of a graph G is defined as follows.

$$T(G,x,y) = \sum_{A \subseteq E} (x-1)^{k(A)-k(E)}(y-1)^{k(A)+|A|-|V|},$$

where $k(A)$ denotes the number of connected components of the graph (V,A).

In statistical physics one often studies the following form of the Tutte polynomial:

$$Z_G(q,v) = \sum_{A \subseteq E} q^{k(A)} v^{|A|}.$$

The two forms are essentially equivalent:

$$T(G,x,y) = (x-1)^{-k(E)}(y-1)^{-|V|} Z_G((x-1)(y-1), y-1).$$

Both forms have several advantages. For instance, it is easy to generalize the latter one to define the multivarite Tutte-polynomial. Let us assign a variable v_e to each edge and set

$$Z_G(q,\underline{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in E} v_e.$$

Note that the chromatic polynomial of graph G is

$$ch(G,x) = Z_G(x,-1) = (-1)^{|V|-k(G)} x^{k(G)} T(G,1-x,0).$$

The main result of this section is the following.

Theorem 2.1. *Let G be a simple graph and let $P(G,x)$ be the co-adjoint polynomial, $T(G,x,y)$ be the Tutte polynomial of the graph G then*

$$P(G,x) = \frac{1}{2^{|V|}} Z(2x,-2) = (-1)^{|V|-k(G)} x^{k(G)} T(G,1-x,-1).$$

Remark 2.2. It is known that the Tutte polynomial satisfies the following recursive formulas:

$$T(G,x,y) = T(G-e,x,y) + T(G/e,x,y)$$

if e is neither a loop nor a bridge and

$$T(G,x,y) = xT(G-e,x,y)$$

if e is a bridge and

$$T(G,x,y) = yT(G/e,x,y)$$

if e is a loop.

This formulas provide a straightforward way to prove Theorem 2.1 by induction. We will not follow this route since whenever we use these recursive formulas we have to distinguish some cases according to the edge being a bridge or not. After some steps the proof would split into too many cases. Instead we use the simple form provided by the polynomial $Z_G(q,v)$.

Proof. Let $|V(G)| = n$ and let us write

$$P(G,x) = \sum_{k=0}^{n} r_k(G)x^k$$

and

$$\frac{1}{2^n}Z(2x,-2) = \frac{1}{2^n}\sum_{A\subseteq E}(2x)^{k(A)}(-2)^{|A|} = \sum_{k=1}^{n}\left(\frac{1}{2^{n-k}}\sum_{\substack{A\subseteq E\\ k(A)=k}}(-2)^{|A|}\right)x^k.$$

Set

$$t_k(G) = \frac{1}{2^{n-k}}\sum_{\substack{A\subseteq E\\ k(A)=k}}(-2)^{|A|}.$$

We have to prove that $t_k(G) = r_k(G)$ for all graph G. We prove it by induction on the number of edges of G. If G is the empty graph on n vertices then both polynomials are x^n and we are done. By the recursive formula $P(G,x) = P(G-e,x) - P(G\Delta e,x)$ we have

$$r_k(G) = r_k(G-e) - r_k(G\Delta e)$$

for an arbitrary edge e. Now let us consider $t_k(G)$. Let e be an arbitrary edge. Clearly, in the sum corresponding to $t_k(G)$ the sets A's not containing the edge e contribute $t_k(G-e)$ to the sum. By induction $t_k(G-e) = r_k(G-e)$.

Now let us consider a set A containing the edge e. Then one can consider $A-e$ as a set of edges in G/e for which $k(A-e) = k$ whence we get that these sets contribute a sum $(-1)t_k(G/e)$; note that $|A-e| = |A|-1$, but G/e has only $n-1$ vertices so the divison and multiplication by 2 cancels each other and only the term -1 remains from the term -2. Hence

$$t_k(G) = t_k(G-e) - t_k(G/e).$$

Thus we only need to prove that

$$r_k(G\Delta e) = t_k(G/e).$$

So far we did not use anything about $G\Delta e$. Observe that $G\Delta e$ is nothing else but the graph obtained from G/e by deleting the multiple edges. Let us consider the multiple edges e_1 and e_2. Assume that for some edge set A of G/e not containing e_1, e_2 we have $k(A\cup\{e_1\}) = k$. Then $k(A\cup\{e_2\}) = k(A\cup\{e_1,e_2\}) = k$ as well and they contribute to the sum

$$(-2)^{|A\cup\{e_1\}|}+(-2)^{|A\cup\{e_2\}|}+(-2)^{|A\cup\{e_1,e_2\}|} = (-2)^{|A|}((-2)+(-2)+(-2)^2) = 0.$$

Hence we can delete the multiple edges from G/e without changing the value of $t_k(.)$:

$$t_k(G/e) = t_k(G\Delta e).$$

By induction we have $t_k(G\Delta e) = r_k(G\Delta e)$. Hence

$$r_k(G) = r_k(G-e)-r_k(G\Delta e) = t_k(G-e)-t_k(G\Delta e) = t_k(G-e)-t_k(G/e) = t_k(G).$$

$\square$

Remark 2.3. By the recursive formula

$$P(G,x) = P(G-e,x) - P(G\Delta e,x)$$

it is easy to prove that the coefficients $r_k(G)$ have alternating signs. On the other hand, it is not clear from the expressions $t_k(G)$.

Remark 2.4. A surprising corollary of Theorem 2.1 is that $|P(G,1)| = 0$ or 1 and it is 1 if and only if the graph is Eulerian, i.e., all degrees are even. It follows from the fact that $|T(G,0,-1)|$ counts the nowhere-0 $\mathbb{Z}_2$-flows (note that the flow polynomial is also a specialization of the Tutte polynomial). Since nowhere-0 $\mathbb{Z}_2$-flow is simply a flow taking the value 1 on all edges, this immediately implies the claim.

3. Exponential type graph polynomials

In the introduction we considered four graph polynomials: the matching polynomial, the chromatic polynomial, the adjoint polynomial and our new graph polynomial, the co-adjoint polynomial. Surprisingly, they all belong to a very special class of graph polynomials, the so-called exponential type graph polynomials.

Definition 3.1. We say that the graph polynomial f is of exponential type if for every graph $G = (V(G), E(G))$ we have $f(\emptyset, x) = 1$ and $f(G,x)$ satisfies that

$$\sum_{S \subseteq V(G)} f(G[S],x) f(G[V \setminus S],y) = f(G,x+y).$$

Note that Gus Wiseman [12] calls these graph polynomials *binomial-type*.

One can deduce from the definition that the chromatic polynomial is of exponential type. For the matching polynomial and the adjoint polynomial this follows from Theorem 3.3 below. This is a structure theorem for the exponential type graph polynomials proven in [1]. For the co-adjoint polynomial this is simply the special case of the following much more general statement.

Theorem 3.2. [10] *For the multivariate Tutte-polynomial $Z_G(q,\underline{v})$ we have*

$$\sum_{S \subseteq V(G)} Z_{G[S]}(q_1,\underline{v}) Z_{G[V\setminus S]}(q_2,\underline{v}) = Z_G(q_1+q_2,\underline{v}).$$

The following theorem characterizes exponential type graph polynomials, see Theorem 5.1 of [1].

Theorem 3.3. [1] *Let b be a function from the class of graphs to the complex numbers. Let us define the graph polynomial f_b as follows. Set*

$$a_k(G) = \sum_{\{S_1,S_2,\dots,S_k\} \in \mathcal{P}_k} b(S_1)b(S_2)\dots b(S_k),$$

where the summation goes over the set $\mathcal{P}_k$ of all partitions of $V(G)$ into exactly k non-empty sets. Then let

$$f_b(G,x) = \sum_{k=1}^{n} a_k(G)x^k,$$

where $n = |V(G)|$. Then

(a) For any function b, the graph polynomial $f_b(G,x)$ is of exponential type.

(b) For any exponential type graph polynomial f, there exist a graph function b such that $f(G,x) = f_b(G,x)$. More precisely, if $b(G)$ is the coefficient of x^1 in $f(G,x)$ then $f = f_b$.

Remark 3.4. For the matching polynomial take $b_m(K_1) = 1$ $b_m(K_2) = -1$ and $b_m(H) = 0$ otherwise. For the adjoint polynomial consider $b_h(K_n) = (-1)^{n-1}$ for complete graphs K_n and $b_h(H) = 0$ otherwise. This proves that the matching and the adjoint polynomials are indeed of exponential type.

Remark 3.5. By the method of Alan Sokal [11] one can prove that the root of $P(G,x)$ of largest modulus has absolute value at most KD where D is the largest degree and

$$K = \inf_a \frac{a + e^a}{\log(1 + ae^{-a})} \approx 7.96.$$

Alan Sokal proved this statement for the chromatic polynomial. On the other hand, one can simply copy his argument to prove this statement for the co-adjoint polynomial. Alternatively, Theorem 1.6 of [1] or the paper [6] provide a weaker, but still linear bound.

4. Complete graphs and balanced complete bipartite graphs

In this section we give the co-adjoint polynomial of some small graphs.

$$P(K_1,x) = x$$

$$P(K_2,x) = x^2 - x$$

$$P(K_3,x) = x^3 - 3x^2 + x$$

$$P(K_4,x) = x^4 - 6x^3 + 7x^2 - 2x$$

$$P(K_5,x) = x^5 - 10x^4 + 25x^3 - 20x^2 + 5x$$

$$P(K_6,x) = x^6 - 15x^5 + 65x^4 - 105x^3 + 70x^2 - 16x$$

$$P(K_7,x) = x^7 - 21x^6 + 140x^5 - 385x^4 + 490x^3 - 287x^2 + 61x$$

$$P(K_8,x) = x^8 - 28x^7 + 266x^6 - 1120x^5 + 2345x^4 - 2548x^3 + 1356x^2 - 272x$$

Clearly, the coefficient of the the term x^1 in $P(G,x)$ is $(-1)^{|V|-1}T(K_n,1,-1)$ by Theorem 2.1. It is known that $a_n = T(K_n,1,-1)$ counts the number of alternating permutations on $n+1$ elements. Let $(-1)^n P(K_n,-x) = p_n(x)$. The graph polynomial $P(G,x)$ is of exponential type, applying this observation to the complete graphs we obtain that

$$\sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) = p_n(x+y).$$

Hence the polynomials $(p_k(x))$ are binomial-type and we know that

$$\sum_{n=0}^{\infty} p_n(x)\frac{z^n}{n!} = \exp(xF(z)),$$

where

$$F(z) = \sum_{n=1}^{\infty} a_n\frac{z^n}{n!}.$$

The exponential generating functions of the alternating polynomials is known, we only need to integrate it since the coefficients are translated:

$$F(z) = \int \frac{1+\sin z}{\cos z} = \ln\frac{1+\sin z}{\cos^2 z}.$$

For balanced complete bipartite graphs we have

$$P(K_{1,1},x) = x^2 - x$$

$$P(K_{2,2},x) = x^4 - 4x^3 + 6x^2 - 2x$$

$$P(K_{3,3},x) = x^6 - 9x^5 + 36x^4 - 66x^3 + 51x^2 - 13x$$

$$P(K_{4,4},x) = x^8 - 16x^7 + 120x^6 - 488x^5 + 1112x^4 - 1360x^3 + 808x^2 - 176x$$

$$P(K_{5,5},x) = x^{10} - 25x^9 + 300x^8 - 2100x^7 + 9150x^6 - 25030x^5 +$$

$$+42020x^4 - 41020x^3 + 20785x^2 - 4081x$$

The sequence of the coefficients of x^1 seems to be very interesting. Note that not only these numbers are $1, 2, 13, 176, 4081, \dots,$ but the values of $P(K_{n,n}, -1)$ are also these numbers. The same phenomenon occurs at $(-1)^n P(K_n, -1)$ and the coefficients of $P(K_{n+2}, x)$. In fact, these are known results. The latter one is quite well-known, but both results are very special cases of the main result of [4] which asserts that under some condition

$$T(G, 1, -1) = T(G - \{u, v\}, 2, -1).$$

Although we do not give the condition of their theorem here, but we note that the complete graphs and complete bipartite graphs satisfy the conditions if (u, v) is an edge.

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