A NOTE ON CHARACTER SUMS

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Abstract. We will investigate certain character sums. We will prove some discrepancy-type inequalities for incomplete sums.

1. Introduction.

We will investigate incomplete sums, in particular we will estimate incomplete character sums. The main result in this direction is Vinogradov's theorem. Throughout this paper we will write $e(\alpha) = e^{2\pi i\alpha}$.

Theorem (Vinogradov [3]): Let q, x, y be positive integers, $0 < x < y \leq q$. Let $a_1, a_2, \dots, a_q \in \mathbb{C}$ and

$$F(t) = \sum_{j=1}^{q} a_j e\left(\frac{jt}{q}\right),$$

Further let $A = \sum_{j=1}^{q} a_j = F(0)$. Then

$$\left|\sum_{n=x}^{y} a_n - \frac{y-x+1}{q}A\right| \leq \frac{1}{2q}\sum_{l=1}^{q-1}\frac{|F(l)|}{||\frac{l}{q}||}$$

where $||x|| = \min_{n\in\mathbb{Z}} |x-n|$.

A consequence of this theorem is the famous Pólya-Vinogradov inequality, which states that if χ is a non-principal character mod q then for any positive integer n

$$\left|\sum_{k=1}^{n} \chi(k)\right| \ll \sqrt{q}\log q.$$

In this paper we will prove a lower bound for incomplete sums in terms of the $|F(l)|$'s and we will apply it for some character sums. In what follows let χ be a primitive character modulo q. Let $S_n = S_n(\chi) = \sum_{k=1}^{n} \chi(k)$ and $L_m = L_m(\chi) = \sum_{n=1}^{m} S_n(\chi)$. We will prove the following theorems.

Theorem 1.1. *Let $a_1, a_2, \dots, a_q$ be complex numbers, $A_k = \sum_{j=1}^{k} a_j$, $A = A_q$. Let $F(l) = \sum_{j=1}^{q} a_j e\left(\frac{jl}{q}\right)$. Then for $1 \leq l \leq q-1$ we have*

$$F(l) = \left(1 - e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}\left(A_j - \frac{j}{q}A\right)e\left(\frac{jl}{q}\right)$$

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and

$$\frac{1}{2\pi}\max_{1\le l\le q-1}\left(\frac{|F(l)|}{||\frac{l}{q}||}\right)\le\sum_{k=1}^{q}\left|A_k-\frac{k}{q}A\right|.$$

We will also prove a variant of this theorem concerning intervals of given length.

Theorem 1.2. *Let* $1\le k\le (q-1)/2$ *be a fixed positive integer, let* $a_1,\dots,a_q,C$ *be complex numbers and* $B_i=\sum_{j=i+1}^{i+k}a_j$*. Then we have*

$$\frac{2}{\pi}\left|\sum_{j=1}^{q}a_je\left(\frac{j}{q}\right)\right|\le\frac{1}{k}\sum_{i=1}^{q}|B_i-C|.$$

We will prove a variant of the Pólya-Vinogradov inequality concerning $L_m(\chi)$.

Theorem 1.3. *For every primitive character* χ *mod* q *there exist a constant* c *and a complex number* $C_q=C_q(\bar{\chi})$ *for which* $\left|L_m(\chi)+\frac{C_qm}{\tau(\bar{\chi})}\right|\le cq^{3/2}$ *for all* m *where* $\tau(\chi)$ *is the Gaussian sum*

$$\tau(\chi)=\sum_{n=1}^{q}\chi(n)e\left(\frac{n}{q}\right).$$

As an application of Theorem 1.1 we will show

Theorem 1.4. *For every primitive character* χ mod q *there exist an* n *and an* m *such that* $1\le n,m\le q$ *and* $|S_n|\ge\frac{1}{2\pi}\sqrt{q}$ *and* $|L_m+\frac{C_qm}{\tau(\bar{\chi})}|\ge\frac{1}{4\pi^2}q^{3/2}$.

Remark: The first statement of Theorem 1.4 is known [1],[2].

2. Proofs of the theorems

Proof of Theorem 1.1. By partial summation we get:

$$\sum_{j=1}^{q}a_je\left(\frac{jl}{q}\right)=\sum_{j=1}^{q}(A_j-A_{j-1})e\left(\frac{jl}{q}\right)=$$

$$=\sum_{j=1}^{q}A_j\left(e\left(\frac{jl}{q}\right)-e\left(\frac{(j+1)l}{q}\right)\right)+A_qe\left(\frac{l}{q}\right)=$$

$$=\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}A_je\left(\frac{jl}{q}\right)+A_qe\left(\frac{l}{q}\right)=$$

$$=\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}\left(A_j-\frac{j}{q}A\right)e\left(\frac{jl}{q}\right)$$

$$+A_q e\left(\frac{l}{q}\right)+\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}\frac{j}{q}Ae\left(\frac{jl}{q}\right).$$

Now we will show that for $1\le l\le q-1$ we have

$$A_q e\left(\frac{l}{q}\right)+\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}\frac{j}{q}Ae\left(\frac{jl}{q}\right)=0.$$

Indeed, let $k_0=0$ and $k_n=e\left(\frac{l}{q}\right)+\cdots+e\left(\frac{nl}{q}\right)=\frac{1-e\left(\frac{nl}{q}\right)}{1-e\left(\frac{l}{q}\right)}e\left(\frac{l}{q}\right)$; then $k_q=0$ since $1\le l\le q-1$. Then we have

$$\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}\frac{j}{q}Ae\left(\frac{jl}{q}\right)$$

$$=\frac{A}{q}\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}je\left(\frac{jl}{q}\right)$$

$$=\frac{A}{q}\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}j(k_j-k_{j-1})=-\frac{A}{q}\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}k_j=$$

$$=-\frac{A}{q}e\left(\frac{l}{q}\right)\sum_{j=1}^{q}\left(1-e\left(\frac{jl}{q}\right)\right)=-Ae\left(\frac{l}{q}\right).$$

Thus

$$F(l)=\left(1-e\left(\frac{l}{q}\right)\right)\sum_{j=1}^{q}\left(A_j-\frac{j}{q}A\right)e\left(\frac{jl}{q}\right).$$

Since $\left|1-e\left(\frac{l}{q}\right)\right|\le 2\pi||\frac{l}{q}||$ it follows that

$$\frac{1}{2\pi}\frac{|F(l)|}{||\frac{l}{q}||}\le\sum_{j=1}^{q}\left|A_j-\frac{j}{q}A\right|.$$

□

Proof of Theorem 1.2. The proof is very similar to the previous one:

$$\sum_{i=1}^{q}(B_i-C)e\left(\frac{i}{q}\right)=\sum_{i=1}^{q}B_ie\left(\frac{i}{q}\right)=$$

$$=\sum_{i=1}^{q}a_i\left(e\left(\frac{i}{q}\right)+\cdots+e\left(\frac{i-k+1}{q}\right)\right)$$

$$=\sum_{i=1}^{q}a_ie\left(\frac{i-k+1}{q}\right)\frac{e\left(\frac{k}{q}\right)-1}{e\left(\frac{1}{q}\right)-1}=$$

$$=e\left(\frac{-k+1}{q}\right)\frac{1-e\left(\frac{k}{q}\right)}{1-e\left(\frac{1}{q}\right)}\sum_{i=1}^{q}a_ie\left(\frac{i}{q}\right).$$

Since $\left|1-e\left(\frac{1}{q}\right)\right| \leq \frac{2\pi}{q}$ and $\left|1-e\left(\frac{k}{q}\right)\right| \geq \frac{4k}{q}$ we have

$$\frac{2}{\pi}\left|\sum_{j=1}^{q} a_j e\left(\frac{j}{q}\right)\right| \leq \frac{1}{k}\sum_{i=1}^{q}|B_i - C|.$$

□

Proof of Theorem 1.3. First we start from the identity

$$\chi(k)=\frac{1}{\tau(\bar{\chi})}\sum_{l=1}^{q}\bar{\chi}(l)e\left(\frac{kl}{q}\right).$$

Then we have

$$\sum_{k=1}^{n}\chi(k)=\sum_{k=1}^{n}\frac{1}{\tau(\bar{\chi})}\sum_{l=1}^{q}\bar{\chi}(l)e\left(\frac{kl}{q}\right)=\frac{1}{\tau(\bar{\chi})}\sum_{l=1}^{q}\bar{\chi}(l)\sum_{k=1}^{n}e\left(\frac{kl}{q}\right)=$$

$$=\frac{1}{\tau(\bar{\chi})}\sum_{l=1}^{q}\bar{\chi}(l)\frac{e\left(\frac{nl}{q}\right)-1}{e\left(\frac{l}{q}\right)-1}e\left(\frac{l}{q}\right).$$

It follows that

$$\sum_{n=1}^{m}\sum_{k=1}^{n}\chi(k)=\frac{1}{\tau(\bar{\chi})}\sum_{l=1}^{q}\bar{\chi}(l)\frac{e\left(\frac{l}{q}\right)}{e\left(\frac{l}{q}\right)-1}\sum_{n=1}^{m}\left(e\left(\frac{nl}{q}\right)-1\right)=$$

$$=\frac{1}{\tau(\bar{\chi})}\sum_{l=1}^{q}\bar{\chi}(l)\frac{e\left(\frac{l}{q}\right)}{e\left(\frac{l}{q}\right)-1}\left(\frac{e\left(\frac{ml}{q}\right)-1}{e\left(\frac{l}{q}\right)-1}e\left(\frac{l}{q}\right)-m\right)=$$

$$=\frac{1}{\tau(\bar{\chi})}\sum_{l=1}^{q}\bar{\chi}(l)\frac{e\left(\frac{2l}{q}\right)}{\left(e\left(\frac{l}{q}\right)-1\right)^2}\left(e\left(\frac{ml}{q}\right)-1\right)-\frac{C_q m}{\tau(\bar{\chi})}$$

where

$$C_q=\sum_{l=1}^{q}\bar{\chi}(l)\frac{e\left(\frac{l}{q}\right)}{e\left(\frac{l}{q}\right)-1}.$$

Hence

$$\left|\sum_{n=1}^{m}\sum_{k=1}^{n}\chi(k)+\frac{C_q m}{\tau(\bar{\chi})}\right|\leq\frac{1}{\sqrt{q}}\sum_{l=1}^{q-1}\frac{2}{16||\frac{l}{q}||^2}\leq\frac{1}{2}q^{3/2}.$$

□

Proposition:

$$\sum_{k=1}^{q}k\chi(k)=\frac{q}{\tau(\bar{\chi})}\sum_{l=1}^{q}\bar{\chi}(l)\frac{e\left(\frac{l}{q}\right)}{e\left(\frac{l}{q}\right)-1}.$$

In other words

$$C_q = \frac{\tau(\bar{\chi})}{q}\sum_{k=1}^{q} k\chi(k).$$

Proof. Let us use the previous theorem with $m = tq$ where t is a large positive integer. Since $\sum_{k=1}^{q}\chi(k) = 0$ we have

$$\left|\sum_{n=1}^{tq}\sum_{k=1}^{n}\chi(k) + \frac{C_q tq}{\tau(\bar{\chi})}\right| = \left|t\sum_{n=1}^{q}\sum_{k=1}^{n}\chi(k) + \frac{C_q tq}{\tau(\bar{\chi})}\right| =$$

$$= \left|t\left(\sum_{k=1}^{q}(q-k+1)\chi(k) + \frac{C_q q}{\tau(\bar{\chi})}\right)\right| = \left|-t\left(\sum_{k=1}^{q}k\chi(k) - \frac{C_q q}{\tau(\bar{\chi})}\right)\right| \le cq^{3/2}.$$

Since t can be arbitrarily large we get

$$\sum_{k=1}^{q} k\chi(k) = \frac{C_q q}{\tau(\bar{\chi})}.$$

□

Remark: We also could have proved this result directly.
If $\chi(-1) = 1$ then one can easily see that $C_q = 0$.

Proof of Theorem 1.4. We start out from the Gaussian sum $\tau(\chi)$.
Let us apply Theorem 1.1. Note that $S_q = 0$, thus

$$\tau(\chi) = \left(1 - e\left(\frac{1}{q}\right)\right)\sum_{n=1}^{q} S_n e\left(\frac{n}{q}\right)$$

Now we have

$$\sqrt{q} = |\tau(\chi)| \le \left|1 - e\left(\frac{1}{q}\right)\right|\sum_{n=1}^{q}|S_n| \le \frac{2\pi}{q}\sum_{n=1}^{q}|S_n|.$$

Thus there exists an n for which $|S_n| \ge \frac{1}{2\pi}\sqrt{q}$, which proves the first statement. To prove the second statement we apply Theorem 1.1 to the sequence $a_n = S_n$; in this case $A_m = L_m$ and

$$\frac{n}{q}A = \frac{q}{n}\sum_{k=1}^{q}(q-k+1)\chi(k) = -\frac{q}{n}\sum_{k=1}^{q}k\chi(k) = -\frac{q}{n}\frac{C_q q}{\tau(\bar{\chi})} = \frac{C_q n}{\tau(\bar{\chi})}$$

by the Propositititon. Thus we can apply Theorem 1.1 to obtain

$$\tau(\chi) = \left(1 - e\left(\frac{1}{q}\right)\right)\sum_{n=1}^{q} S_n e\left(\frac{n}{q}\right) =$$

$$= \left(1 - e\left(\frac{1}{q}\right)\right)^2 \sum_{n=1}^{q}\left(L_n + \frac{C_q n}{\tau(\bar{\chi})}\right) e\left(\frac{n}{q}\right).$$

Hence

$$\sqrt{q} = |\tau(\chi)| \le \left|1 - e\left(\frac{1}{q}\right)\right|^2 \sum_{n=1}^{q} \left|L_n + \frac{C_q n}{\tau(\bar{\chi})}\right| \le$$
$$\le \left(\frac{2\pi}{q}\right)^2 \sum_{n=1}^{q} \left|L_n + \frac{C_q n}{\tau(\bar{\chi})}\right|.$$

Thus there exists an n for which $|L_n + \frac{C_q n}{\tau(\bar{\chi})}| \ge \frac{1}{4\pi^2} q^{3/2}$.

□

As a consequence of it we show that for some $1 \le m \le q$ we have $|L_m| > cq^{3/2}$ with some positive absolute constant c, which with the Pólya-Vinogradov theorem shows that at least $c\frac{q}{\log q}$ n's of the interval $[1, q]$ we have $|S_n| \gg q^{1/2}$.

Proposition: For some $1 \le m \le q$ we have $|L_m| > cq^{3/2}$ with some positive absolute constant c.

Proof. We will prove the proposition with $c = \frac{1}{8\pi^2}$.

If $\chi(-1) = 1$ then this is a trivial consequence of Theorem 1.4 since in this case $C_q = 0$. It is also trivial if $|\sum_{n=1}^{q} S_n(\chi)| \ge \frac{1}{8\pi^2} q^{3/2}$ since we choose $m = q$. Finally if $|\sum_{n=1}^{q} S_n(\chi)| \le \frac{1}{8\pi^2} q^{3/2}$ then by Theorem 1.4 for some $1 \le m \le q$ we have

$$\left|\sum_{n=1}^{m} S_n(\chi)\right| \ge \left|\sum_{n=1}^{m} S_n(\chi) - \frac{m}{q}\sum_{n=1}^{q} S_n(\chi)\right| - \left|\frac{m}{q}\sum_{n=1}^{q} S_n(\chi)\right|$$
$$\ge \frac{1}{4\pi^2} q^{3/2} - \frac{1}{8\pi^2} q^{3/2} = \frac{1}{8\pi^2} q^{3/2}.$$

□

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