

ON A CONJECTURE OF V. NIKIFOROV

PÉTER CSIKVÁRI

ABSTRACT. In this paper we will prove that

$$\mu(G) + \mu(\overline{G}) \leq \frac{1 + \sqrt{3}}{2}n - 1$$

where $\mu(G), \mu(\overline{G})$ are the greatest eigenvalues of the adjacency matrices of the graph G and its complement and n denotes the number of vertices of G .

1. INTRODUCTION

In [6] Nosal proved that if G is a simple graph on n vertices then

$$\mu(G) + \mu(\overline{G}) \leq \sqrt{2}n.$$

Later Nikiforov showed [5] that $\sqrt{2}$ is not the best constant in this inequality. He proved

$$\mu(G) + \mu(\overline{G}) \leq (\sqrt{2} - \varepsilon)n$$

where $\varepsilon = 8 \cdot 10^{-7}$. In the same paper Nikiforov conjectured that

$$\mu(G) + \mu(\overline{G}) \leq \frac{4}{3}n + O(1).$$

The constant $4/3$ cannot be decreased since the clique of size $\frac{2}{3}n$ as graph G (or its complement) attains this bound.

Nikiforov also proved [4] that it is meaningful to search for the best constant; in general he proved that if $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G)$ are the eigenvalues of the adjacency matrix of graph G of order n , \overline{G} is the complement of G and $F(G)$ is a fixed linear combination of $\mu_i(G)$, $\mu_{n-i+1}(G)$, $\mu_i(\overline{G})$ and $\mu_{n-i+1}(\overline{G})$ ($1 \leq i \leq k$), then the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \max\{F(G) : |V(G)| = n\}$$

always exists.

Although we could not prove Nikiforov's conjecture we could improve on the constant $\sqrt{2} - \varepsilon$ significantly. We will prove that

Theorem 1.1. *Let G be a simple graph on n vertices. Then*

$$\mu(G) + \mu(\overline{G}) \leq \frac{1 + \sqrt{3}}{2}n \leq 1.3661n$$

We will also prove that $o(n)$ error terms can be ignored in our upper bounds.

Key words and phrases. Spectral radius; Eigenvalue; Nordhaus-Gaddum type problem.

Part of this work was carried out while the author was the guest of the University of Memphis.

Theorem 1.2. *Assume that the bound*

$$\mu(G) + \mu(\overline{G}) \leq cn + o(n)$$

holds for all graph G with some constant c and $|V(G)| = n$. Then for all graphs on n vertices we have

$$\mu(G) + \mu(\overline{G}) \leq cn - 1.$$

The two theorems together give the result stated in the abstract.

The structure of this paper is the following. In the next section we will prove a theorem on Kelmans's operation; Kelmans's operation will be the main tool in the proof of Theorem 1.1. In Section 3 we will prove an upper bound on the spectral radius of the threshold graphs of the Kelmans operation. In this section we also deduce Theorem 1.1 from earlier results concerning the Kelmans's operation and its threshold graphs. In Section 4 we will prove Theorem 1.2.

Notation: We will follow the usual notation: G is a graph, $V(G)$ is the set of its vertices, $e(G)$ denotes the number of edges, $N(x)$ is the set of the neighbors of x , $\mu = \mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ are the eigenvalues of the adjacency matrix of G , μ is also called spectral radius. $|N(v_i)| = \deg(v_i) = d_i$ denote the degree of the vertex v_i .

2. KELMANS'S OPERATION AND ITS THRESHOLD GRAPHS

In [3] Kelmans introduced and studied the following transformation. Let u, v be two vertices of the graph G , we obtain the Kelmans transformed of G as follows: we erase all edges between u and $N(u) \setminus (N(v) \cup \{v\})$ and add all edges between v and $N(u) \setminus (N(v) \cup \{v\})$. Let us call v the *beneficiary* of the transformation. The obtained graph has the same number of edges as G . Kelmans showed that this transformation decreases the number of spanning trees of the graph G . More generally Kelmans's transformation turned out to be an efficient tool in the theory of reliable networks [1], [8]. Now we prove that this transformation increases the spectral radius of G and \overline{G} at the same time!

Theorem 2.1. *Let G be a graph and G' be a graph obtained from G by some Kelmans's transformation. Then*

$$\mu(G') \geq \mu(G) \quad \text{and} \quad \mu(\overline{G'}) \geq \mu(\overline{G})$$

Proof. The key observation is that up to isomorphism G' is independent of u or v being the beneficiary if we apply the transformation to u and v . Indeed, in G' one of u or v will be adjacent to $N_G(u) \cup N_G(v)$, the other will be adjacent to $N_G(u) \cap N_G(v)$ (and if the two vertices are adjacent in G then they will remain adjacent also).

Let \vec{x} be the non-negative eigenvector of unit length belonging to $\mu(G)$ and let A_G and $A_{G'}$ be the corresponding adjacency matrices. Assume that $x_u \geq x_v$ and choose u to be the beneficiary of the Kelmans transformation. By the above observation, this choice does not affect the resulting graph G' .

Then

$$\begin{aligned} \mu(A_{G'}) &= \max_{\|\vec{y}\|=1} \vec{y}^T A_{G'} \vec{y} \geq \vec{x}^T A_{G'} \vec{x} = \\ &= \vec{x}^T A_G \vec{x} + 2(x_u - x_v) \sum_{w \in N(u) \setminus (N(v) \cup \{v\})} x_w \geq \mu(A_G) \end{aligned}$$

This way we have proved the first statement. The second statement follows from the first statement since a Kelmans transformation is also a Kelmans's transformation to the complement of the graph; although the role of the beneficiary and the other vertex change. \square

Now we determine the threshold graphs of this transformation. Let us say that u dominates v if $N(v) \setminus \{u\} \subset N(u) \setminus \{v\}$. Clearly, if we apply Kelmans's transformation to graph G and u and v such that u is the beneficiary then u will dominate v in G' .

Theorem 2.2. *By some application of Kelmans's transformation one can always transform an arbitrary graph G to a graph G_{tr} which satisfies the following condition. The vertices of G_{tr} can be ordered such way that whenever $i < j$ then v_i dominates v_j .*

Proof Let $d_1(G) \geq d_2(G) \geq \dots \geq d_n(G)$ be the degree sequence of graph G . One can define a lexicographic ordering: let us say that $G_1 \succ G_2$ if for some k $d_k(G_1) > d_k(G_2)$ and $d_i(G_1) = d_i(G_2)$ for $1 \leq i \leq k-1$. Those graphs which have the same degree sequence cannot be distinguished by this ordering, but this will not be a problem for us.

Now let us choose the graph G^* which can be attained by some application of Kelmans's transformation from G and in the lexicographic ordering is one of the best among these graphs. We show that this graph has the desired property. Indeed if $\deg_{G^*}(v_i) \geq \deg_{G^*}(v_j)$, but v_i does not dominate v_j then one can apply a Kelmans's transformation to G^* and v_i and v_j where v_i is the beneficiary; then in the obtained graph the degree of v_i is strictly greater than $\deg(v_i)$ thus the obtained graph is better in the lexicographic ordering than G^* contradicting the choice of G^* . \square

Remark 2.3. From now on we refer to the threshold graphs of the Kelmans transformation as threshold graphs.

Remark 2.4. These graphs, or more precisely their adjacency matrices appear in the article of Brualdi and Hoffman [2]. Rowlinson called these matrices stepwise matrices [7].

3. UPPER BOUND TO THE SPECTRAL RADIUS OF THRESHOLD GRAPHS

We prove a simple upper bound on the spectral radius of graphs belonging to a certain class of graphs. We will see later that this class contains the threshold graphs of the Kelmans transformation.

Theorem 3.1. *Let us assume that in the graph G the set $X = \{v_1, v_2, \dots, v_k\}$ forms a clique while $V \setminus X = \{v_{k+1}, \dots, v_n\}$ forms an independent set. Furthermore let $e(X, V \setminus X)$ denote the number of edges going between X and*

$V \setminus X$. Then

$$\mu(G) \leq \frac{k-1 + \sqrt{(k-1)^2 + 4e(X, V \setminus X)}}{2}$$

Proof. We can assume that G is not the empty graph, for which the statement is trivial. Let \vec{x} be the eigenvector belonging to $\mu = \mu(G)$. For $1 \leq j \leq k$ we have

$$\mu x_j = x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_k + \sum_{v_m \in N(v_j) \cap V \setminus X} x_m$$

By adding together these equations we get

$$\mu \left(\sum_{j=1}^k x_j \right) = (k-1) \left(\sum_{j=1}^k x_j \right) + d_{k+1}x_{k+1} + \cdots + d_n x_n$$

For $k+1 \leq j \leq n$ we have

$$\mu x_j = \sum_{v_m \in N(v_j)} x_m$$

Since $V \setminus X$ forms an independent set we have $\mu x_j \leq \sum_{i=1}^k x_i$ for $k+1 \leq j \leq n$ and so

$$\begin{aligned} \mu \left(\sum_{j=1}^k x_j \right) &= (k-1) \left(\sum_{j=1}^k x_j \right) + d_{k+1}x_{k+1} + \cdots + d_n x_n \leq \\ &\leq (k-1) \left(\sum_{j=1}^k x_j \right) + \frac{d_{k+1}}{\mu} \left(\sum_{j=1}^k x_j \right) + \cdots + \frac{d_n}{\mu} \left(\sum_{j=1}^k x_j \right) \end{aligned}$$

Since $\sum_{j=k+1}^n d_j = e(X, V \setminus X)$ we have

$$\mu \leq k-1 + \frac{e(X, V \setminus X)}{\mu}$$

Hence

$$\mu(G) \leq \frac{k-1 + \sqrt{(k-1)^2 + 4e(X, V \setminus X)}}{2}$$

□

Remark 3.2. Let G be a threshold graph for which v_i dominates v_j whenever $i < j$. Let k be the least integer for which v_k and v_{k+1} are not adjacent. In this case $X = \{v_1, \dots, v_k\}$ forms a clique while $V \setminus X = \{v_{k+1}, \dots, v_n\}$ forms an independent set. One can prove a bit stronger inequalities for the threshold graphs:

$$\frac{1}{\mu} \left(\sum_{j=k+1}^n d_j^2 \right) \leq k\mu - k(k-1)$$

and

$$\mu^2 + \mu \leq k(k-1) + \frac{1}{\mu} \left(\sum_{j=k+1}^n d_j^2 \right) + e(X, V \setminus X)$$

By combining these inequalities we immediately get the statement of the theorem.

Remark 3.3. For our purposes the inequality

$$\mu(G) \leq \frac{k + \sqrt{k^2 + 4e(X, V \setminus X)}}{2}$$

will suffice.

Proof of Theorem 1.1. By Theorem 2.1 and 2.2 we only need to check the statement for threshold graphs. Let G be a threshold graph for which v_i dominates v_j whenever $i < j$. Let k be the least integer for which v_k and v_{k+1} are not adjacent. In this case $X = \{v_1, \dots, v_k\}$ forms a clique while $V \setminus X = \{v_{k+1}, \dots, v_n\}$ forms an independent set. Let us apply Theorem 3.1 with G and X and with \overline{G} and $V \setminus X$. Then we have

$$\mu(G) \leq \frac{k + \sqrt{k^2 + 4e_G(X, V \setminus X)}}{2}$$

and

$$\mu(\overline{G}) \leq \frac{n - k + \sqrt{(n - k)^2 + 4e_{\overline{G}}(V \setminus X, X)}}{2}$$

Thus we have

$$2(\mu(G) + \mu(\overline{G})) - n \leq \sqrt{k^2 + 4e_G(X, V \setminus X)} + \sqrt{(n - k)^2 + 4e_{\overline{G}}(V \setminus X, X)}$$

By the arithmetic-square mean inequality we have

$$\begin{aligned} & \sqrt{k^2 + 4e_G(X, V \setminus X)} + \sqrt{(n - k)^2 + 4e_{\overline{G}}(V \setminus X, X)} \leq \\ & \leq \sqrt{2(k^2 + 4e_G(X, V \setminus X) + (n - k)^2 + 4e_{\overline{G}}(V \setminus X, X))} = \\ & = \sqrt{2(k^2 + (n - k)^2 + 4k(n - k))} \leq \sqrt{3}n \end{aligned}$$

Altogether we get

$$2(\mu(G) + \mu(\overline{G})) - n \leq \sqrt{3}n$$

Hence

$$\mu(G) + \mu(\overline{G}) \leq \frac{1 + \sqrt{3}}{2}n$$

□

4. ERROR TERM

In this section we prove Theorem 1.2. The argument we apply here is essentially due to V. Nikiforov.

For any graph $G = (V, E)$ and integer $t \geq 1$, write $G^{(t)}$ for the graph obtained by replacing each vertex $u \in V$ by a set V_u of t independent vertices and joining $x \in V_u$ to $y \in V_v$ if and only if $uv \in E$. Further let $G^{[t]} = \overline{G^{(t)}}$, i.e., $G^{[t]}$ is obtained from $G^{(t)}$ by joining all vertices within V_u for every $u \in V$. It is easy to prove the following two facts:

(i) The eigenvalues of $G^{(t)}$ are $t\mu_1(G), \dots, t\mu_n(G)$ together with $n(t - 1)$ additional 0's.

(ii) The eigenvalues of $G^{[t]}$ are $t\mu_1(G) + t - 1, \dots, t\mu_n(G) + t - 1$ together with $n(t - 1)$ additional (-1) 's.

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Assume that we know that for all graph G we have

$$\mu(G) + \mu(\overline{G}) \leq cn + o(n)$$

where c is an absolute constant and n denotes the order of G .

Now let us apply it to $G^{(t)}$ then we have

$$\mu(G^{(t)}) + \mu(\overline{G^{(t)}}) \leq ctn + o(tn)$$

Now we have $\mu(G^{(t)}) = t\mu(G)$ and $\mu(\overline{G^{(t)}}) = \mu(\overline{G}^{[t]}) = t\mu(\overline{G}) + t - 1$. Thus

$$t\mu(G) + t\mu(\overline{G}) + t - 1 \leq ctn + o(tn)$$

After dividing by t we have

$$\mu(G) + \mu(\overline{G}) + 1 - \frac{1}{t} \leq cn + \frac{o(tn)}{t}$$

Let us consider n to be fix and t going to infinity then we get

$$\mu(G) + \mu(\overline{G}) + 1 \leq cn$$

□

Remark 4.1. This argument together with Kelmans's operation shows that the

$$\sup_G \frac{\mu(G) + \mu(\overline{G}) + 1}{n}$$

cannot be attained. Indeed if G attained the supremum then so would $G^{(t)}$, but $G^{(t)}$ cannot be the threshold graph of the Kelmans operation if it is not the empty graph, so one can apply Kelmans's transformation to it gaining a strictly better graph. (Of course the empty graph cannot attain the supremum, because the supremum is at least $\frac{4}{3}$ while the corresponding value to the empty graph is exactly 1.)

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DEPARTMENT OF ALGEBRA AND NUMBER THEORY, EÖTVÖS LORÁND UNIVERSITY,
BUDAPEST H-1117, HUNGARY
E-mail address: csiki@cs.elte.hu